b
UNIVERSITÄT
BERN

Faculty of Business, Economics
and Social Sciences

Department of Economics

Estimating the Welfare Costs of Very High Inflations and Hyperinflations

Luca Benati, Juan-Pablo Nicolini

24-01

February, 2024

DISCUSSION PAPERS

Schanzeneckstrasse 1
CH-3012 Bern, Switzerland
http://www.vwi.unibe.ch

Estimating the Welfare Costs of Very High Inflations and Hyperinflations

Luca Benati
University of Bern*

Juan-Pablo Nicolini
Federal Reserve Bank
of Minneapolis and Universidad Di Tella†

Abstract

We explore the welfare costs of inflation originating from lack of liquidity satiation for Weimar Republic's hyperinflation and three high-inflation countries. Towards the peak of Weimar's hyperinflation the costs are estimated to have been equal to nearly 20 per cent of income. For Israel, Mexico, and Argentina the costs had been materially lower, but still in the range of 3-5 per cent of GDP.

*Department of Economics, University of Bern, Schanzeneckstrasse 1, CH-3001, Bern, Switzerland. Email: luca.benati@vwi.unibe.ch

†Federal Reserve Bank of Minneapolis, 90 Hennepin Avenue, Minneapolis, MN 55401, United States. Email: juanpa@minneapolisfed.org

1 Introduction

As it has been extensively documented, very high inflations have uniformly been associated with macroeconomic mayhem and the destruction of wealth held in nominal assets. Evidence is especially stark for hyperinflations. For Weimar Republic's episode, for example, the data reported in Table XL of Graham (1930, p. 317) show that the unemployment rate among trade union members, which in 1922 had oscillated between 0.6 and 3.3 per cent, increased rapidly following the invasion of the Ruhr on the part of France in January 1923, which, as pointed out by Bresciani-Turroni (1937), '*gave the coup de grâce to the national finances and the German mark*', thus inaugurating the final and most extreme phase of the hyperinflation. Unemployment reached 6.2 per cent in May, 9.9 in September, and it further increased to a remarkable 28.2 per cent in December, the last month of the hyperinflation.

Rather than further exploring such well known and well documented aspects of very high inflations and hyperinflations, in this note we narrowly focus on an issue which, to the very best of our knowledge, has received virtually no previous attention, i.e. the welfare costs of these episodes originating from lack of liquidity satiation along the lines of the classic work of Bailey (1956), Friedman (1969), Lucas (2000), and Ireland (2009). Our main finding is tha for very high inflations and hyperinflations these costs are far from negligible, as at the inflation peaks of the respective episodes they range between more than 3 per cent of output for Mexico; about 4.5 per cent for Argentina and Israel; and between 18 and 20 per cent for Weimar's hyperinflation.

The note is organized as follows. The next section outlines the theoretical framework underlying our estimates of the welfare costs of inflation. Section 3 discusses the data and their statistical properties. Section 4 presents and discusses our estimates.

2 Theory

Since the very start the macroeconomic literature on money demand has been dominated by two alternative functional forms, Cagan's (1956) *semi-log*

$$\ln\left(\frac{M_t}{Y_t}\right) = \ln(A) - \xi r_t \tag{1}$$

and Meltzer's (1963) *log-log*,

$$\ln\left(\frac{M_t}{Y_t}\right) = \ln(B) - \eta \ln(r_t) \tag{2}$$

where M_t, r_t, and Y_t are the nominal money stock, the opportunity cost of money, and nominal GDP, respectively; η and ξ are the elasticity and semi-elasticity of money demand, respectively; and A and B are constants. Building upon Benati, Lucas, Nicolini, and Weber (2021), Benati and Nicolini (2024) derive (1) and (2) as the solutions to the problem of a representative agent who, in each period, has to choose the optimal number of portfolio transactions (n_t) that allow her to exchange interest-bearing illiquid assets for money, which she needs in order to buy the consumption good. Benati and Nicolini's (2024) analysis is reported in full in Online Appendix A to the present note. In this section we only report and discuss its main features and results.

2.1 A model of money demand

The representative agent has standard preferences (e.g., her utility function is differentiable, increasing and concave). The total cost of transactions, measured in units of time, is given by a function $\theta(n_t, \nu_t)$, where ν_t is an exogenous stochastic process. This formulation generalizes the linear functional form assumed by Baumol (1952) and Tobin (1956). The production technology for the consumption good is given by $y_t = c_t = z_t l_t$ where l_t is time devoted to the production of the final consumption good and z_t is an exogenous stochastic process.

In each period the representative agent is endowed with one unit of time that is used to produce goods and to make transactions. Thus, equilibrium in the labor market implies that $1 = l_t + \theta(n_t, \nu_t)$ and feasibility is given by $c_t = z_t(1 - \theta(n_t, \nu_t))$. Purchases are subject to a cash in advance constraint $P_t c_t \leq n_t M_t$, where M_t are average money balances and n_t is the number of portfolio adjustments within each period. The variable n_t is the only economically relevant decision to be made by the representative agent.

At the beginning of each period the agent starts with nominal wealth W_t, that can be allocated to money or interest bearing bonds, B_t so a restriction to the optimal problem of the agent is $M_t + B_t \leq W_t$. Nominal wealth at the beginning of next period will then be given by

$$W_{t+1} \leq M_t(1 + r_t^m) + B_t(1 + r_t^b) + T_t + [1 - \theta(n_t, \nu_t)]\, z_t P_t - P_t c_t$$

where r_t^b is the return on government bonds, r_t^m=0 is the return on money, and T_t is a transfer made by the monetary authority. Notice that the unconstrained efficient outcome is to allocate all the labor input to the production of the consumption good so as to set $c_t = z_t$: thus, in equilibrium a measure of the welfare cost of making transactions—i.e., the welfare costs of inflation—as a fraction of consumption, is given by the value of $\theta(n_t, \nu_t)$.

Benati and Nicolini (2024) show[1] that the solution to the representative agent problem is given by

$$n_t^2 \frac{\theta_n(n_t, \nu_t)}{(1 - \theta(n_t, \nu_t))} = r_t^b \tag{A.13}$$

The specific functional form taken by the demand for real money balances depends on the transactions technology $\theta(n_t, \nu_t)$. If $\theta(n_t, \nu_t) = \gamma \nu_t n_t^\sigma$, for γ, σ>0, then money

[1] See Section A.2 of the Online Appendix to the present note.

demand takes the log-log functional form. On the other hand, if

$$\theta(n_t, \nu_t) = -b\frac{\ln(\varepsilon + n_t)}{n_t + \varepsilon} - \frac{k + \nu_t}{n_t + \varepsilon} + \left(b\frac{\ln\varepsilon}{\varepsilon} + \frac{k + \nu_t}{\varepsilon}\right),$$

for parameters b, k, ε >0, the demand for real money balances takes the semi-log form.

In what follows we will exclusively focus on the log-log functional form, and we will instead eschew the semi-log. The reason for this is that, as shown by Benati, Lucas, Nicolini, and Weber (2021) and especially Benati (2024), for high-inflation countries and hyperinflation episodes the data exhibit a clear preference for the log-log specification.[2]

2.2 The welfare costs of inflation

Abstracting from the stochastic process ν_t and eliminating time dependence, the welfare cost of inflation measured as a fraction of consumption is given by

$$\omega^W(r) = \theta(n(r)), \text{ where } \omega^W(0) = \theta(n(0)) = 0.$$

Benati and Nicolini (2024) use theoretical results due to Alvarez, Lippi, and Robatto (2019) to compute lower and upper bounds for $\omega^W(r)$. As we report in Online Appendices A.4 and A.5 to the present note, the bounds are given by

$$\frac{\omega^D(r)}{(1 + \omega^D(r))} \leq \omega^W(r) \leq \omega^D(r), \tag{3}$$

with the function $\omega^D(r)$ being given, for Meltzer's (1963) log-log, by

$$\omega^D_{\log - \log}(r) = \ln(B)\frac{\eta}{1 - \eta}r^{1-\eta}, \tag{4}$$

Based on an estimated money demand curve the expression for $\omega^D(r)$ immediately allows to compute lower and upper bounds for the welfare costs of inflation. An important point to stress is that these bounds hold for *any* value of the opportunity cost of money, and therefore also for the very high inflation and hyperinflationary episodes we analyze herein.

3 The Data

For the Weimar Republic's hyperinflation, monthly data on the velocity of circulation of money based on wholesale prices are from Table XXII of Bresciani-Turroni (1937).

[2] See in particular Benati's (2024) Figures 7 and 8, which visually compare the semi-log and log-log specifications, and his Section 6.3, which reports the results of a likelihood-based model comparison exercise for the two functional forms.

The series had been normalized by 1913 (i.e. for the year 1913 it took a value of one). Based on Benati, Lucas, Nicolini and Weber's (2021) data, however, in 1913 German money velocity had been equal to 7.49. Consequently, we have rescaled Bresciani-Turroni's money velocity series by multiplying it by 7.49.[3] A series for the inflation rate is from Cagan (1956). A series for the money market rate ('Tägliches Geld') is from Table 23 of Holtfrerich (1980). The sample period is September 1920-October 1923.

For Israel, quarterly seasonally adjusted data on nominal GDP and the CPI are from the Central Bureau of Statistics, whereas a series for M1 is from Israel's central bank. A series for the Treasury bill rate is from the International Monetary Fund's International Financial Statistics. The sample period is 1981Q2-2019Q4, thus excluding the COVID pandemic.

For Mexico, quarterly seasonally adjusted data on nominal GDP are from Mexico's statistical agency, INEGI. Quarterly seasonally adjusted data for the CPI, M1, and a 3-month government bond yield are all from the Banco de México.

For Argentina, quarterly seasonally adjusted data on real GDP (since 1980Q1), nominal GDP (since 2004Q2), M1 and the CPI are all from the Banco Central de la República Argentina (Argentina's central bank). Since nominal GDP is only available since 2004Q2, in order to compute M1 velocity we have proceeded as follows. Since 2004Q2 we simply take the ratio between nominal GDP and nominal M1. For the period 1980Q1-2004Q1 we take the ratio between real GDP and real M1 (deflated by the CPI), and we rescale it in such a way that for 2004Q2 the resulting series takes the same value as the ratio between nominal GDP and nominal M1.

Whereas, strictly speaking, neither Israel nor Mexico ever experienced a hyperinflation (which, following Cagan (1956), is routinely defined as an episode during which the monthly log-difference of the price level exceeds 0.5), Argentina did, between January 1987 and April 1991. Since the large monthly spikes in the monthly inflation rate that characterize hyperinflations tend to be somehow 'sveraged out' when going to lower frequencies, especially for Argentina (but also for Israel and Mexico) it would have been of interest to work with monthly data. Unfortunately, in spite of extensive searches we were unable to locate data that allowed us to compute monthly series for money velocity for these three countries, and we were therefore compelled to work with quarterly data.

For all four countries, in what follows we work with money velocity (i.e., the inverse of money balances as a fraction of GDP), and a series for the opportunity cost of money, which we compute as the maximu, at each point in time, between inflation and the series for the nominal short-term interest rate (for Argentina we were not able to find an interest rate series, and we therefore work with inflation).

[3] In fact, working with Bresciani-Turroni's original series produces manifestly absurd results, with the welfare costs of inflation even taking values in excess of 100 per cent of GDP.

3.1 Evidence from unit roots and cointegration tests

For all countries, Elliot, Rothenberg, and Stock (1996) tests bootstrapped as in Diebold and Chen (1996) do not allow to reject the null of a unit root in the logarithms of either money velocity or the opportunity cost of money.[4] One possible interpretation of these results is that for all countries both series feature *exact* unit roots. A more plausible interpretation is that they are *near* unit root, which in small samples are statistically indistinguishable from exact unir root processes. This is the case in particular for interest rates, for which a direct implcation of them featuring an exact unit root—i.e., that they could literally take *any* value between minus and plus infinity—appears as manifestly absurd.

Assuming that the series are near unit root processes, we therefore proceed to test for cointegration based on Wright's (2000) test, which was designed to be equally valid for both exact and near unit root processes. In brief, for all four countries Wright's (2000) tests, bootstrapped according to the procedure proposed by Benati, Lucas, Nicolini and Weber (2021), do not reject the null hypothesis of cointegration between the logarithm of the opportunity cost and the logarithm of velocity (i.e., minus the logarithm of money balances as a fraction of GDP). Finally, Hansen and Johansen's (1999) tests for stability in the cointegration vector do not detect any evidence of instability.[5]

In what follows we will therefore proceed under the assumption the two series are cointegrated, and we will compute the lower and upper bounds for the welfare costs of inflation detailed in (3)-(4) based on estimated log-log money demand specifications.

4 The Welfare Costs of High Inflations and Hyperinflations

For all countries we estimate Meltzer's (1963) log-log functional form

$$\ln V_t = \ln(B) - \eta \ln(r_t) + \xi_t \tag{5}$$

where V_t is velocity, and ξ_t is a regression residual, based on Stock and Watson's (1993) dynamic OLS (DOLS) procedure. This produces point estimates of $\ln(B)$ and η, which based on (3)-(4) allow to compute point estimates of the lower and upper bounds of the welfare costs of inflation.

We characterize uncertainty around the point estimates as in Benati and Nicolini (2024). Specifically, following Luetkepohl (1991, pp. 370-371), we estimate the VECM for the two series via OLS by imposing in estimation the cointegration vector that we previously estimated via Stock and Watson's (1993) DOLS procedure. We

[4] We do not report these results for reasons of space, but they are available upon request.

[5] Again, we do not report this evidence for reasons of space, but it is available upon request.

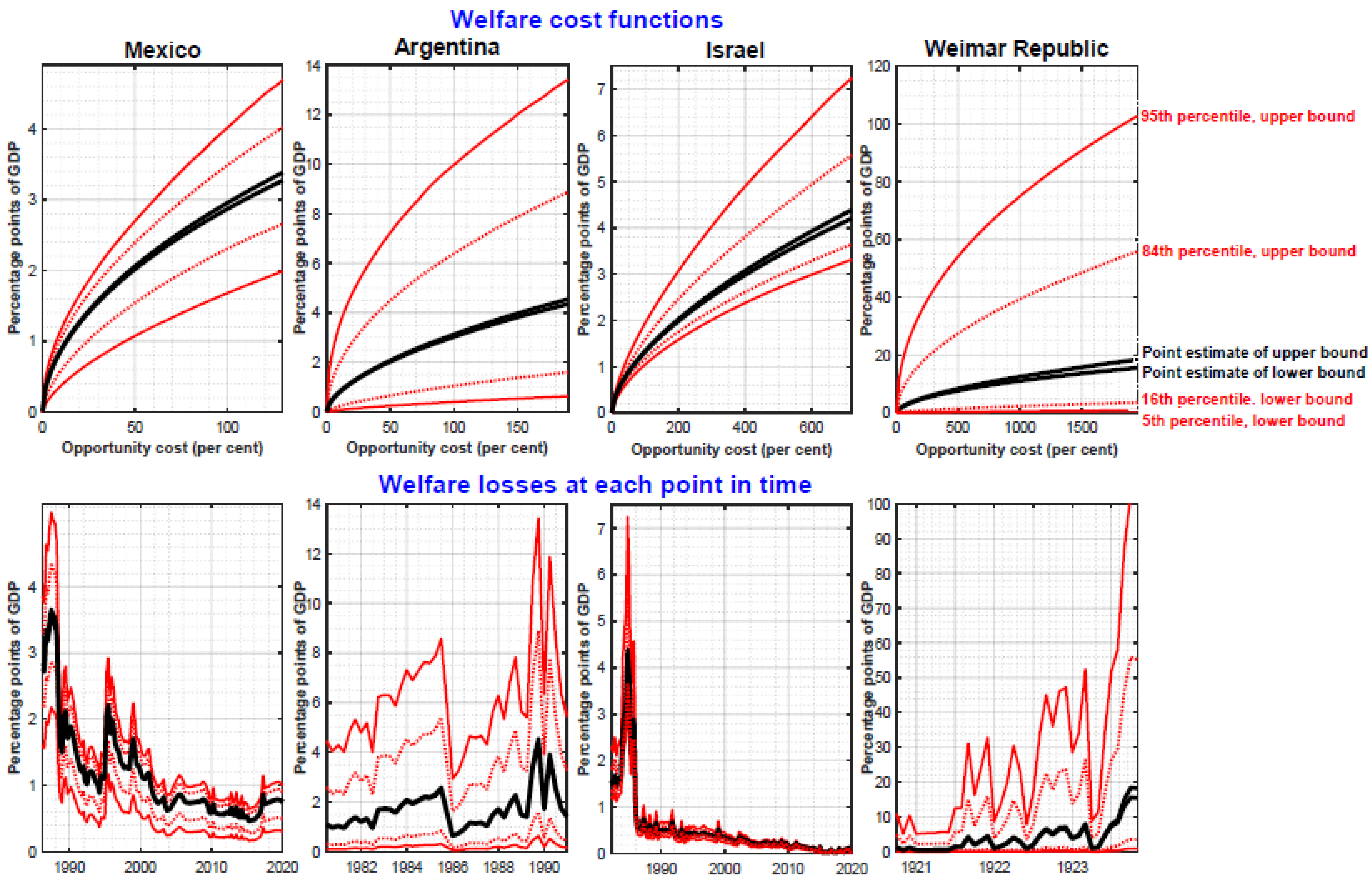

Figure 1 Estimated welfare cost functions and welfare losses at each point in time

then bootstrap the VECM as in Cavaliere *et al.* (2012), thus characterizing uncertainty about the point estimates of the relevant objects (i.e., the lower and upper bounds of the welfare cost functions and the welfare losses at each point in time).

In line with the previous discussion in Section 3.1, this procedure is valid if the series contain exact unit roots. Under the alternative possible interpretation of the results from unit root tests, i.e. that the series are *local-to-unity*, we proceed as in Benati *et al.* (2021, Section 4.2.1). Specifically, we compute, based on the just-mentioned VECM, the corresponding VAR in levels, which by construction features one, and only one exact unit root, and we turn it into its corresponding near unit root VAR by shrinking the unit root to λ=1-0.5$\cdot$(1/T), where T is the sample length.[6] Finally, we characterize uncertainty about the point estimates by bootstrapping such near unit root VAR. In short, the two bootstrapping procedures produce numerically near-identical results. We report evidence based on the near unit root VAR, but the alternative set of results is available upon request.

Figure 1 reports the results. The top row shows the estimated welfare cost functions (in percentage points of GDP) for values of the opportunity cost of money from zero to the maximum value that it had taken over the sample period. The bottom row shows the estimated welfare losses at each point in time. The thick black lines are the point estimates of the lower and upper bounds, whereas the red lines are the 84th and 95th percentiles of the bootstrapped distribution of the upper bound, and the 5th and 16th percentiles of the corresponding distribution of the lower bound.

Several facts clearly emerge from the figure:

first, in line with Benati and Nicolini's (2024) evidence for low-inflation countries, based on the simple point estimates the upper and lower bounds of both the welfare cost functions, and the welfare losses at each point in time are quite remarkably tight, to the point that in a few instances (in particular, the welfare losses in the second row) they are near-uniformly indistinguishable.

On the other hand, *second*, with the single exception of the welfare losses at each point in time for Israel, uncertainty is uniformly substantial, sometimes quite remarkably so. This is the case in particular for the welfare cost functions for Argentina and the Weimar's Republic. In the former case, at the peak of the opportunity cost the 90 per cent-coverage bootstrapped confidence interval ranges between 0.5 and 13.5 per cent. In the latter case, the width of the corresponding interval is about 100 percentage points. By the same token, in the second half of 1923 the estimated welfare losses for the Weimar Republic's hyperinflation consistently range from nearly zero to in excess of 50 per cent, and in September-October nearly 100 per cent. Evidence for Mexico and Israel is less dramatic, but uncertainty is still quite large: e.g., for Mexico the 90 per cent-coverage confidence interval for the welfare cost function at the peak of the opoprtunity cost ranges between 2 and 4.7 per cent.

Third, and most importantly, at the peaks of the inflation episodes the welfare costs had been uniformly sizeable, ranging (based on point estimates) from about 3.7

[6] For details see Benati *et al.*'s (2021) footnote 24.

to about 4.4 per cent of GDP for Mexico, Argentina, and Israel, to 16-19 per cent of output in September-October 1923 during Weimar's hyperinflation. This provides a stark illustration of how, beyond the already well known and widely documented costs of very high inflations and hyperinflations (in terms of economic mayhem, and the destruction of wealth held in nominal assets), they have consistently imposed non-negligible, and sometimes large costs uniquely in terms of lack of liquidity satiation, by compelling agents to hold comparatively low levels of real money balances.

5 References

Alvarez, F, F. Lippi, and R. Robatto (2019): "Cost of Inflation in Inventory Theoretical Models", *Review of Economic Dynamics*, vol. 32, pages 206-226, April.

Baumol, W. J. (1952): "The Transactions Demand for Cash: An Inventory Theoretic Model", *Quarterly Journal of Economics*, 66, 545-372.

Bailey, M. J. (1956): "The Welfare Cost of Inflationary Finance", *Journal of Political Economy*, 64(2), 93-110.

Belongia, M.T., and P.N. Ireland (2019): "The Demand for Divisia Money: Theory and Evidence", *mimeo*, May 2019

Benati, L.(2024): "The Monetary Dynamics of Hyperinflation Reconsidered", see at: https://www.nber.org/conferences/si-2023-monetary-economics

Benati, L., R. E. Lucas Jr., J.P. Nicolini, and W. Weber (2021): "International Evidence on Long-Run Money Demand", *Journal of Monetary Economics*, 117, 43-63

Benati, L., and J.P. Nicolini (2024): "The Welfare Costs of Inflation", University of Bern and Federal Reserve Bank of Minneapolis, mimeo

Bresciani-Turroni, C. (1937): *The Economics of Inflation*, John Dickens and Co. Northampton.

Graham, F. D. (1930): *Exchange, Prices, and Production in Hyperinflation*, Princeton University Press.

Holtfrerich, C.L. (1980): *Die Deutsche Inflation 1914-1923: Ursachen und Folgen in Internationaler Perspektive.* Berlin/New York: Walter de Gruyter, S. 70. Primärquelle: Die Wirtschaftskurve 1922, 1923 passim.

Cavaliere, G., A. Rahbek, and A. M. R. Taylor (2012): "Bootstrap Determination of the Cointegration Rank in Vector Autoregressive Models", *Econometrica*, 80(4), 1721-1740.

Diebold, F. X., and C. Chen (1996): "Testing Structural Stability with Endogenous Breakpoint: A Size Comparison of Analytic and Bootstrap Procedures", *Journal of Econometrics*, 70(1), 221-241.

Elliot, G., T. J. Rothenberg, and J. H. Stock (1996): "Efficient Tests for an Autoregressive Unit Root", *Econometrica*, 64(4), 813-836.

Ireland, P. (2009): "On the Welfare Cost of Inflation and the Recent Behavior of Money Demand", *American Economic Review*, 99(3), 1040-1052.

Lucas Jr., R.E. (2000): "Inflation and Welfare", *Econometrica*, 68(2), 247-274.

Luetkepohl, H. (1991): *Introduction to Multiple Time Series Analysis*, 2nd edition. Springer-Verlag.

Sidrauski, M. (1967*a*): "Inflation and Economic Growth", *Journal of Political Economy*, 75(6), 796-810.

Sidrauski, M. (1967*b*): "Rational Choice and Patterns of Growth in a Monetary Economy", *American Economic Review*, 57(2), 534-544.

Stock, J. H., and M. W. Watson (1993): "A Simple Estimator of Cointegrating Vectors in Higher Order Integrated Systems", *Econometrica*, 61(4), 783-820.

Tobin, J. (1956): “The Interest Elasticity of Transactions Demand for Money”, *Review of Economics and Statistics*, 38, 241-247.

Wright, J.H. (2000): “Confidence Sets for Cointegrating Coefficients Based on Stationarity Tests”, *Journal of Business and Economic Statistics*, 18(2), 211-222 .

Online Appendix for: Estimating the Welfare Costs of Very High Inflations and Hyperinflations

Luca Benati
University of Bern*

Juan-Pablo Nicolini
Federal Reserve Bank of Minneapolis and Universidad Di Tella†

*Department of Economics, University of Bern, Schanzeneckstrasse 1, CH-3001, Bern, Switzerland. Email: luca.benati@vwi.unibe.ch

†Federal Reserve Bank of Minneapolis, 90 Hennepin Avenue, Minneapolis, MN 55401, United States. Email: juanpa@minneapolisfed.org

A Computing the lower and upper bounds for the welfare costs of inflation

A.1 The theoretical model

We study a labor-only economy with uncertainty in which making transactions is costly.[1] The economy is inhabited by a unit mass of identical agents with preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{A.1}$$

where U is differentiable, increasing and concave.

Every period, the representative agent chooses a number of portfolio transactions n_t that allow her to exchange interest-bearing illiquid assets for money, that is needed to buy the consumption good. The total cost of those transactions, measured in units of times, is given by a function $\theta(n_t, \nu_t)$, where ν_t is an exogenous stochastic process. This formulation generalizes the linear function assumed by Baumol (1952) and Tobin (1956).

The production technology for the consumption good is given by

$$y_t = c_t = z_t l_t$$

where l_t is time devoted to the production of the final consumption good and z_t is an exogenous stochastic process.

The representative agent is endowed, each period, with a unit of time that is used to produce goods and to make transactions. Thus, equilibrium in the labor market implies that

$$1 = l_t + \theta(n_t, \nu_t)$$

and feasibility is given by

$$c_t = z_t(1 - \theta(n_t, \nu_t)).$$

It follows that the real wage is equal to z_t.

Purchases are subject to a cash in advance constraint

$$P_t c_t \leq n_t M_t \tag{A.2}$$

where M_t are average money balances and n_t is the number of portfolio adjustments within each period. The variable n_t is the only economically relevant decision to be made by the representative agent. In line with the literature we set the nominal return on money to $r_t^m = 0$.

[1] The baseline model is discussed at length in Benati et. al. (2021).

At the beginning of each period, the agent starts with nominal wealth W_t, that can be allocated to money or interest bearing bonds, B_t so a restriction to the optimal problem of the agent is

$$M_t + B_t \leq W_t, \tag{A.3}$$

Nominal wealth at the beginning of next period, in state s_{t+1}, will then be given by

$$\begin{aligned} W_{t+1} \leq \; & M_t(1+r_t^m) + B_t(1+r_t^b) + T_t \\ & + [1-\theta(n_t,\nu_t)]\, z_t P_t - P_t c_t \end{aligned} \tag{A.4}$$

where r_t^b is the return on government bonds and T_t is a transfer made by the monetary authority.

Notice that the unconstrained efficient outcome is to allocate all the labor input to the production of the consumption good so as to set $c_t = z_{t.}$ Thus, a measure of the welfare cost of making transactions, as a fraction of consumption, is given by the value of $\theta(n_t, \nu_t)$ in equilibrium.

A.2 Model solution

The problem of the agent is to maximize (A.1) by choosing c_t, n_t, M_t, B_t, and W_{t+1} subject to (A.2)-(A.4). Assume that the function $\theta(n_t, \nu_t)$ is differentiable. If we let ξ_t, λ_t and δ_t be the corresponding Lagrange multipliers, the first order conditions are given by

$$\beta^t U(c_t) = P_t\lambda_t + P_t\delta_t \tag{A.5}$$

$$P_t\lambda_t\theta_n(n_t,\nu_t)z_t = M_t\delta_t \tag{A.6}$$

$$\xi_t = \lambda_t(1+r_t^m) + \delta_t n_t \tag{A.7}$$

$$\xi_t = \lambda_t(1+r_t^b) \tag{A.8}$$

$$\lambda_t = E_t\xi_{t+1} \tag{A.9}$$

The first-order conditions imply that, as long as $r_t^b > r_t^m = 0$.

$$\lambda_t = \frac{\delta_t n_t}{r_t^b - r_t^m} \tag{A.10}$$

and from this we obtain

$$P_t \frac{\delta_t n_t}{r_t^b - r_t^m}\theta_n(n_t,\nu_t)z_t = M_t\delta_t \tag{A.11}$$

or

$$\frac{n_t}{r_t^b - r_t^m}\theta_n(n_t,\nu_t)z_t = \frac{M_t}{P_t} \tag{A.12}$$

Note also that , as long as $r_t^b > r_t^m = 0$. it ought to be the case that $\delta_t > 0$, which means that the cash-in-advance constraint is binding, so

$$\frac{M_t}{P_t c_t} = \frac{1}{n_t},$$

so real money demand, as a proportion of consumption, is equal to the inverse of n_t. Together with feasibility

$$c_t = z_t(1 - \theta(n_t, \nu_t)),$$

this implies

$$\frac{z_t(1 - \theta(n_t, \nu_t))}{n_t} = \frac{M_t}{P_t}$$

Replacing on (A.12) above

$$n_t^2 \frac{\theta_n(n_t, \nu_t)}{(1 - \theta(n_t, \nu_t))} = r_t^b - r_t^m \tag{A.13}$$

Thus, the solution for n_t depends only on the two stochastic processes $r_t^b - r_t^m$ and ν_t. Note, in particular, that it does not depend on z_t, so the theory implies a unit income elasticity.

Note that, in general, the solution for n_t, and therefore the solution for real money demand, depends on the interest rate *differential* between bonds and money, $r_t \equiv r_t^b - r_t^m$. Since, in line with the literature, we assume that $r_t^m = 0$, real money demand here depends on $r_t \equiv r_t^b$.

For the maximum problem of the agent to be well defined, it has to be the case that

$$r_t \geq 0. \tag{A.14}$$

which is the well-known lower bound on the interest rates in bonds.[2]

A.3 Alternative functional forms for the demand for real money balances

The functional form of the demand for real money balances depends on the transactions technology $\theta(n_t, \nu_t)$, and at this level of generality the model is consistent with many different possibilities. In what follows we consider three well-known functional forms that have been used in previous empirical work. All of the three functional forms exhibit a unit income elasticity, as implied by the model. The first specification is the log-log one,

$$\ln \frac{M_t}{P_t y_t} = a^1 - \eta \ln r_t + u_t^1, \tag{A.15}$$

[2] Intuitively, where $r(s^t) - r^m(s^t)$ to be negative, the representative agent would have incentives to borrow from the government unbounded quantities and hold money.

that exhibits a constant interest rate elasticity equal to η. Notice that as $i_t \to 0$. real money demand goes to infinity. The other two formulations are the semi-log

$$\ln \frac{M_t}{P_t y_t} = a^2 - \gamma r_t + u_t^2, \tag{A.16}$$

that exhibits a constant semi-elasticity, γ, and the Selden-Latané

$$\frac{M_t}{P_t y_t} = \frac{1}{a^3 + \phi r_t + u_t^3}. \tag{A.17}$$

Both of them imply a finite level of the demand for real money balances when the r_t becomes zero. As we show below, the welfare costs implications of the last two functional forms are similar.

In the next sub-section we show how to build tight upper and lower bounds for the welfare cost of inflation, using the area under the estimated real money demand function.

A.4 The welfare cost of inflation and the area under the money demand curve

In this sub-section, we apply the techniques developed in Alvarez, Lippi and Robatto (2019) to a class of models that is more restrictive than the ones they used. Specifically, we only consider representative agent models in which the cost of transforming liquid into illiquid assets is given by the differentiable function $\theta(n_t, \nu_t)$ described above. For this restricted class of models we obtain upper and lower bounds for the welfare cost of inflation that can be directly computed based on estimated money demand functions.

Alvarez, Lippi and Robatto (2019) show that the area under the money demand curve approximates the welfare cost of inflation arbitrarily well as the opportunity cost of money (in our model, r_t) approaches zero.[3] An important point to stress is that our bounds can be used for *any* value of the opportunity cost of money, and therefore also for the very high inflation and hyperinflationary episodes we analyze in this note.

In order to make progress and to simplify the notation we eliminate the shock and the time dependence, and we write (A.13) as

$$n^2 \frac{\theta_n(n)}{(1 - \theta(n))} = r, \tag{A.18}$$

where $r \geq 0$. As previously discussed, the welfare cost of inflation, measured as a fraction of consumption, is given by

$$\omega^W(r) = \theta(n(r)), \text{ where } \omega^W(0) = \theta(n(0)) = 0.$$

[3] They also show in numerical examples that the approximation is remarkably accurate for a wide range of positive values of the opportunity cost.

It follows that

$$\frac{\partial \omega^W(r)}{\partial r} = \frac{\partial \theta(n)}{\partial n}\frac{\partial n}{\partial r}(r) > 0. \tag{A.19}$$

We now show how the function $\omega^W(r)$ can be bounded above and below using the integral under the money demand curve.[4]

The area under the demand curve is equal to

$$\omega^D(r) = \int_0^r m(z)dz - m(r)r, \tag{A.20}$$

so

$$\frac{\partial \omega^D(r)}{\partial r} = -\frac{\partial m}{\partial r}(r)r > 0.$$

As real money demand $m(r)$ is the inverse of velocity, $n(r)$, it follows that

$$\frac{\partial n}{\partial r}(r) = -\frac{\partial m}{\partial r}(r)n^2$$

which, using (A.18), becomes

$$\frac{\partial n}{\partial r}(r) = -\frac{\partial m}{\partial r}(r)r\frac{[1-\theta(n)]}{\frac{\partial \theta(n)}{\partial n}}.$$

Using the definition in (A.19),

$$\frac{\partial \omega^W(r)}{\partial r} = -\frac{\partial m}{\partial r}(r)r\,[1-\theta(n)] = \frac{\partial \omega^D(r)}{\partial r}\left[1-\omega^W(r)\right]$$

Recall that $\omega^W(0) = \omega^D(0) = 0$. Thus, we can recover the welfare cost of inflation for an interest rate differential r_0 by integrating $\partial\omega^W/\partial r$ from zero to r_0, or

$$\int_0^{r_0} \frac{\partial \omega^W(z)}{\partial r}dz = \int_0^{r_0} \frac{\partial \omega^D(r)}{\partial r}\left[1-\omega^W(z)\right]dz$$

For all $z \in [0.r_0]$, however,

$$1 \geq \left[1-\omega^W(z)\right] \geq \left[1-\omega^W(r_0)\right]$$

Therefore

$$\int_0^{r_0} \frac{\partial \omega^W(z)}{\partial r}dz \leq \int_0^{r_0} \frac{\partial \omega^D(r)}{\partial r}dz$$

and

$$\int_0^{r_0} \frac{\partial \omega^W(z)}{\partial r}dz \geq \left[1-\omega^W(r_0)\right]\int_0^{r_0} \frac{\partial \omega^D(r)}{\partial r}dz$$

[4]The analysis below follows closely the ideas in Alvarez, Lippi and Robatto (2019).

which imply

$$\left[1-\omega^{W}(r_0)\right]\omega^{D}(r_0) \leq \omega^{W}(r_0) \leq \omega^{D}(r_0)$$

We therefore obtain our bounds as

$$\frac{\omega^{D}(R)}{(1+\omega^{D}(R))} \leq \omega^{W}(R) \leq \omega^{D}(R).$$

Explicit closed form solutions for the function $\omega^{D}(R)$ can be obtained for the three empirical specifications described in (A.15)-(A.17), as we now show.

A.5 Closed form solutions for the function $\omega^{D}(R)$

The previous analysis implies that the parameters of the demand for real money balances, and the lower bound we impose upon the short term interest rate are the only relevant features to compute the welfare costs of inflation in any given country. In order to see this, it is useful to compute the integral under the money demand curve, as defined in (A.18), for the three specifications. The integrals are given by

$$\omega_{\log-\log}(r) = a^{1}\frac{\eta}{1-\eta}r^{1-\eta}, \tag{A.21}$$

$$\omega_{semi-\log}(r) = \frac{a^2}{\gamma}\left(1-\frac{1+\gamma r}{e^{\gamma r}}\right) \tag{A.22}$$

and

$$\omega_{Sel-Lat}(r) = \frac{1}{\phi}\ln\left(\frac{a^3+\phi r}{a^3}\right) - \frac{r}{a^3+\phi r}, \tag{A.23}$$

respectively, for the log-log, the semi-log and the Selden-Latané. As it is apparent, each expression features a slope parameter and a level parameter. These two parameters, together with the assumption regarding the own return on money, fully summarize all of the information that is required for the computation of the welfare costs of inflation.

Based on an estimated money demand curve, and an assumption about the lower bound on nominal interest rates, the expression for the function $\omega^{D}(R)$ immediately allows to compute the welfare costs of inflation.

B References

Alvarez, F, F. Lippi, and R. Robatto (2019): "Cost of Inflation in Inventory Theoretical Models," Review of Economic Dynamics, Elsevier for the Society for Economic Dynamics, vol. 32, pages 206-226, April.

Baumol, W. J. (1952): “The Transactions Demand for Cash: An Inventory Theoretic Model”, *Quarterly Journal of Economics*, 66, 545-372.

Benati, L., R. E. Lucas Jr., J.P. Nicolini, and W. Weber (2021): “International Evidence on Long-Run Money Demand”, *Journal of Monetary Economics*, 117, 43-63

Tobin, J. (1956): “The Interest Elasticity of Transactions Demand for Money”, *Review of Economics and Statistics*, 38, 241-247.