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Monetary Policy in a World of Cryptocurrencies

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Monetary Policy in a World of Cryptocurrencies*

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Abstract

Can currency competition affect central banks' control of interest rates and prices? Yes, it can. In a two-currency world with competing cash (material or digital), the growth rate of the cryptocurrency sets an upper bound on the nominal interest rate and the attainable inflation rate, if the government currency is to retain its role as medium of exchange. In any case, the government has full control of the inflation rate. With an interest-bearing digital currency, equilibria in which government currency loses medium-of-exchange property are ruled out. This benefit comes at the cost of relinquishing control over the inflation rate.

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In recent years cryptocurrencies have attracted the attention of consumers, media and policymakers.[1] Cryptocurrencies are digital currencies, not physically minted. Monetary history offers other examples of uncoined money. For centuries, since Charlemagne, an "imaginary" money existed but served only as unit of account and never as, unlike today's cryptocurrencies, medium of exchange.[2] Nor is the coexistence of multiple currencies within the borders of the same nation a recent phenomenon. Medieval Europe was characterized by the presence of multiple media of exchange of different metallic content.[3] More recently, some nations contended with dollarization or eurization.[4]

However, the landscape in which digital currencies are now emerging is quite peculiar: they have appeared within nations dominated by a single fiat currency just as central banks have succeeded in controlling the value of their currencies and taming inflation.

In this perspective, this article asks whether the presence of multiple currencies can jeopardize the primary function of central banking – controlling prices and inflation – or eventually limit their operational tools – e.g. the interest rate. The short answer is: yes it can.

The analysis posits a simple endowment perfect-foresight monetary economy, in which currency provides liquidity services through cash, either physical or digital. For the benchmark single-currency model, the results are established: the central bank can control the rate of inflation by setting the nominal interest rate; the (initial) price level instead is determined by an appropriate real tax policy.[5] The combination of these two policies (interest-rate targeting and fiscal policy) determines the path of the price level in all periods, and the central bank can achieve any desired inflation rate by setting the right nominal interest rate.

I add to this benchmark a privately issued currency that is perfectly substitutable for the government's currency in providing liquidity services. The private currency is "minted" each period according to a constant non-negative growth rate μ or, alternatively, its supply is reduced by invalidating some tokens.

The first important result is that private currency can be worthless if it is believed to be, while government currency always has a positive value. This result follows from the connection between the policy followed by the government, the levying of taxes in the government currency and the existence of a market of interest-bearing securities

[1] See BIS (2018).

[2] See Einaudi (1936) for an analysis of the "imaginary" money from the time of Charlemagne to the French Revolution. Loyo (2002) studies optimal choice of unit account in a context of multiple units.

[3] Cipolla (1956,1982,1990) describes several cases in the monetary history of coexistence of multiple currencies.

[4] See Calvo and Vegh (1997) for an analysis of dollarization.

[5] Canzoneri, Cumby and Diba (2001), Niepelt (2004), Sims (1994, 2000, 2013), Woodford (1995, 2001, 2003) and Bassetto and Sargent (2020) among others, present results of the benchmark single-currency case.

in it.

The second result is that the perfect control of prices that the government has in the single-currency framework extends unchanged to a multiple-currency model only when $\mu > 0$. Instead, when the supply of private money is constant or decreases over time, there can be equilibria in which the private money circulates as a pure bubble and the government loses control of the price level.

The final result highlights a failure of the price mechanism in not conveying the "correct" information on which is the best money. This failure leads to multiple equilibria. To understand this result, consider the private issuer setting a constant growth rate μ, and the government a gross inflation rate Π through an interest rate policy.[6] If $1 + \mu < \Pi$, there is an equilibrium in which only private money is used as a medium of exchange with a lower inflation rate than government currency, $\Pi^* < \Pi$. However, there is also another equilibrium, with lower welfare, in which both currencies provide liquidity services and have the same inflation rate, $\Pi = \Pi^*$. In this equilibrium, private money becomes less and less important over time in purchasing consumption goods. Gresham's law works in a way that the "bad" money crowds out the "good" money. Although the latter equilibrium is dominated in welfare, the price mechanism does not allow agents to rule out the inferior equilibrium and choose the "best" money. Their choice is only governed by the relative return between the two moneys, which is related to the inflation rates. If households expect the same return, i.e. $\Pi = \Pi^*$, they will hold both moneys irrespective of whether one is "good" or "bad".

The results described above can be helpful to shed light on an important debate between Friedrich von Hayek and Milton Friedman on the desirability of currency competition. Milton Friedman's view is that a purely private system of fiduciary currencies would inevitably lead to price instability (Friedman, 1960). In my model, currency competition does create troubles for the government to control the price level only when the growth rate of private money is non-positive. Friedrich von Hayek's view is instead that unfettered competition in the currency market is beneficial for society (Hayek, 1976). My model contrasts also this view showing that the relative-price mechanism that disciplines agents' choice on which currency to hold does not convey the "correct" information on which is the best money. Therefore, currency competition does not necessarily improve welfare, although it will never worsen it. On the other side, a government that risks to lose the property of medium of exchange for its currency could decide to crowd out private currency to avoid getting into such dangerous territories. In this case, it has to keep inflation, Π, low and bounded by the growth rate of private money, i.e. $\Pi \leq 1+\mu$. In this way, currency competition

[6] In the model presented in Section 1, I consider the possibility of a sunspot shock, which destroys the stock of private money. In the presentation of the results in this introduction, I abstract from it.

could thus become a useful way of keeping inflation low and reducing liquidity premia. However, the inflation bound imposed on the government can be too tight when other objectives, such as economic stabilization, require a higher inflation target.

Next I discuss how results change in an environment in which households can hold deposits at the central bank, which together with other default-free government securities provide liquidity services in competition with an unbacked digital private money. This framework is in line with recent proposals for a central-bank digital currency by allowing the public to hold accounts at the central bank. Results change in a considerable way.

First, there is never an equilibrium in which only the private money is used as a medium of exchange rate. With a central bank digital currency, the government can rule out equilibria in which it loses medium-of-exchange properties for its currency.

The second result is that this benefit comes at the cost of relinquinshing the control over the inflation rate. Unless the government is able to back the supply of interest-bearing liabilities with enough taxes, multiple equilibria still arise depending on the undetermined "realization" of the exchange rate between the two currencies, and the policy followed by the private issuer. The path of the inflation rate is now funcion of the exchange-rate "realization" and of the private-money growth rate, in contrast with the full control that the government had in the case in which cash was the only medium of exchange. A final result, and common to previous analysis, is that currency competition never worsens welfare.

This paper is related to the seminal work of Kareken and Wallace (1981) with two important differences. In their work money is only a store of value, while here it is also a medium of exchange providing liquidity services. Moreover, government sets policy in terms of interest rate and taxes, whereas in their model monetary policies are specified in terms of money growth. These differences explain why the multiplicity of equilibria found in my work is not present in their analysis. First, note that both frameworks share a sort of Gresham's law for which the "good" money, with lower growth rate, is crowded out by the "bad" money, in the case both moneys coexist as medium of exchange. However, in their model, the exchange rate can be any constant value, or zero, or infinite. In my model, instead, the exchange rate can also appreciate over time to support an equilibrium in which the "good" private money is the only used as a medium of exchange, while being at the same time store of value together with the government currency. Therefore, there is an equilibrium in which the "good" and "bad" money are both used as medium of exchange and store of value, with a constant exchange rate between them and with the "good" money vanishing in real terms. There is also an equilibrium in which the "good" money is the only one used as a medium of exchange, with the "bad" money remaining a store of value. In the latter equilibrium, the "good" money appreciates over time.

A recent literature prompted by the increasing number of cryptocurrencies has revived interest in multiple-currency monetary models, which share with my framework a role for money both as store of value and medium of exchange. Fernandez-Villaverde and Sanches (2019) evaluate the role of competing private currencies whose supply is determined by profit maximization.[7] Their results differ substantially from mine. They find that an appropriately defined price stability equilibrium can arise only under certain restrictions on the cost function for private money production. But, when the marginal cost goes to zero, price stability cannot be an equilibrium. Further they find that competition does not achieve the efficient allocation and is socially wasteful.

Schilling and Uhlig (2019) also analyze coexistence and competition between traditional fiat money and cryptocurrencies. But, they are more concerned with the determination of the price of cryptocurrencies, deriving interesting bounds and asset-price relationships. On policy, they assume that the government has always full control of the inflation rate. With respect to monetary policy, they emphasize the connection between the indeterminacy of the cryptocurrency, prices and government money supply.[8]

Marimon et al. (2012) find that currency competition can achieve efficiency relying on a seigniorage function which is maximized at a deflation rate equal to the rate of time preferences. Kovbasyuk (2018), instead, finds that private currency can tend to inflate prices and harm agents who hold fiat currency.

Whereas all the above mentioned works share some similarities in the way of modelling currency competition when cash is the only medium of exchange, none has considered an environment in which government's interest-bearing liabilities can compete with an unbacked private money, to provide liquidity services.

Woodford (1995) is the benchmark reference for the analysis of price determination through interest-rate targeting and fiscal rules in a single-currency economy. He also studies a "free banking" regime in which deposits, in units of the single currency, compete with government money in providing liquidity services. In this case, he finds that the nominal interest rate and the inflation rate are determined by structural factors. However, he does not analyze multiple-currency models. Similarly, Marimon et al. (2003) find that also inside money puts downward pressure on inflation. However, the reason for limitations to policy, in this literature, originates from the way private securities are backed rather than from direct competition for medium-of-exchange properties coming from unbacked private currency, as in my framework.

The paper is structured as follows. Section 1 presents the two-currency model. Section 2 discusses the equilibrium conditions. Section 3 discusses the extension of the model by allowing government securities to provide liquidity services. Section 4

[7] Klein (1974) is an early example of a model of currency competition with profit-maximizing suppliers.

[8] In one-currency models, Sargent and Wallace (1982) and Smith (1988) have studied the coexistence, and competition, between government money and private assets and the need to separate the credit and money markets to avoid fluctuations in the price level unrelated to fundamentals.

concludes.

1 The model

We consider a two-currency economy, one issued by the government and one privately. There are important differences between how we model the two moneys and we start from here. Consider first the private money and its process of creation:

$$M_t^{*p} = T_t^* + \gamma M_{t-1}^{*p},$$

where M_t^{*p} is the supply of private money, T_t^* are transfers in units of private money to households, with $T_t^* \geq 0$, and γ is a parameter, which is, however, a policy choice for the issuer of private currency, with $0 \leq \gamma \leq 1$. In the case $T_t^* > 0$, the private issuer sets $\gamma = 1$. When $\gamma \leq 1$, it is assumed $T_t^* = 0$. Note, first, that transfers from the issuers of private money to households cannot be negative. This is because, unlike the government, the private issuer does not have taxation power.

Consider first the case $T_t^* > 0$ and $\gamma = 1$, the issuer is increasing the supply of its private money by transferring the increase of the new stock to the household. When instead $0 \leq \gamma < 1$, it is assumed $T_t^* = 0$. In this way we model a decrease in the stock of private money, in absence of taxation power. Given its digital form, the issuer can always destroy some tokens by putting out of validity their code – a process that could be validated by the blockchain. In this case, we are assuming that a measure $1 - \gamma$ of tokens runs out of business. This modelling device allows to characterize a negative growth rate for the private money. However, as it will be clear later, the outcome of the process of destruction is different from what would result instead were the reduction of money coming from taxation.[9] In the latter case, the household would still have a stock of money M_{t-1}^* at the beginning of time t, in the former case, instead, the household remains only with a fraction of the money bought in the previous period, i.e. γM_{t-1}^*.

For government money, we assume that government has a taxation power and can issue debt securities at a risk-free rate. Its budget constraint is

$$M_t^g + \frac{B_t^g}{1+i_t} = T_t + M_{t-1}^g + B_{t-1}^g \tag{1}$$

where M_t^g is the supply of cash, and B_t^g is the debt position of the government with respect to default-free interest-bearing securities; i_t is the risk-free nominal interest

[9] One possibility for a private issuer of currency to reduce the stock of money supply, by having a negative transfer T^*, is to use profits from an economic activity. In the context of the model presented in this paper, this can be done by having a fraction of the endowment at disposal of the private issuer, which can then be used to reduce its liabilities (money). Later, we discuss the implications of this assumption for the main results of the paper.

rate, T_t are transfers in units of government currency, if negative it denotes taxes. Government debt, $B_t^g = B_t^T + X_t$, includes treasury's debt, B_t^T, and central bank's reserves, X_t. It is important to specify that the central bank can also issue interest-bearing reserves since I am going to set monetary policy for the central bank in terms of the interest rate on reserves. Note that $X_t \geq 0$ and that the central bank can set at the same time interest on reserves and quantity of reserve.[10] We do not restrict B_t^T to be non-negative, allowing for the treasury to accumulate assets from the private sector, for a given tax policy. This will turn out to be the case in some equilibria, as it will be discussed later.

The model is stochastic and the only source of uncertainty is a sunspot shock for which the private money becomes completely worthless. Denote with h_t the state of nature at time t; h_t can take two values, 0 and 1. When $h_t = 0$ the private money is worthless, and the state is absorbing, so that no private money will be issued anymore. When $h_t = 1$, private money can potentially circulate, but circulation will be an endogenous outcome. Let h^t be the history at time t, defined as $h^t = (h_{t,} h_{t-1}, \ldots . h_{t_0})$; let $h^t = \bar{h}$ denote an history with all states equal to 1, then $\Pr(h_{t+1} = 1 | h^t = \bar{h}) = \alpha$, with $0 \leq \alpha \leq 1$ and therefore $\Pr(h_{t+1} = 0 | h^t = \bar{h}) = 1 - \alpha$. As said earlier, $h_t = 0$ is an absorbing state.

Before presenting the model in detail, I discuss what it can or cannot capture in terms of currency competition and its practical relevance. First, this is a closed-economy model with two currencies: one is a government currency, which could be valued because of taxation power, and the other is a private currency, completely unbacked. The framework is therefore different from the old literature on currency competition in an international context, see Calvo and Vegh (1996) for a comprehensive survey. That literature analyzed competition between different government currencies. The key difference between that framework and mine is that here, for what concerns the competing currency, there is no possibility of taxation, and transfers accrue directly the households living in the domestic economy. Therefore the private currency in this model cannot be interpreted as a foreign government currency, let's say dollars, that competes with a local currency, because in this case transfers or taxes would accrue the foreign economy and not the domestic one, unlike the current framework. Can it be thought as a cryptocurrency? The process of creation of cryptocurrencies is quite convoluted, but the simplification that their growth rate follows some determined path is not far from reality, as well as the absence of taxation power or the possibility to invalidate tokens given their digital nature. There are some noteworthy settlement costs of cryptocurrencies, which we abstract from. The environment described here is then suitable to characterize a limiting case in which all these costs are negligible. See Schilling and Uhlig (2019) for an analysis with transaction costs.

[10] See Woodford (2000).

Let us move now to present the problem of the household. Consider an agent maximizing the expected present-discounted value of utility given by

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ U(C_t) + L\left(\frac{M_t}{P_t} + \frac{S_t M_t^*}{P_t} \right) \right\}, \tag{2}$$

where β is the intertemporal discount factor with $0 < \beta < 1$; the function $U(\cdot)$ is an increasing and concave function of its argument; C is consumption of a perishable good; $L(\cdot)$ is also an increasing and concave function of real money balances in the two currencies. The function has a satiation point at $\bar{m}$, such that $L_m(\cdot) = 0$ for all values of its argument above or equal to $\bar{m}$; M_t and M_t^* are the household's "cash" holdings of the two currencies, P_t is the price of the consumption good in terms of the government currency and S_t is the exchange rate between the two currencies, the amount of government money needed to buy one unit of private money. I am assuming that the two types of cash are perfect substitutes for their liquidity services, when they are both used and when the sunspot shock has not realized, i.e. $h_t = 1$. When instead $h_t = 0$, the exchange rate is zero, i.e. $S_t = 0$, since the private money is worthless. The assumption of linearity between the two moneys simplifies the analysis at little cost in generality. At least, it enables the model to challenge the results that derive from the single-currency framework to a greater extent. The reader should note that the degree of substitution between the two moneys is in this model endogenous depending on the relative price between the two currencies. Indeed, I will discuss equilibria in which one of the two currencies is not used as medium of exchange.

The household is subject to the following budget constraint at time t

$$W_t^h \leq A_t + M_{t-1} + \gamma S_t M_{t-1}^* - P_t C_t + P_t Y + T_t + S_t T_t^*, \tag{3}$$

with

$$W_t^h = E_t \left\{ Q_{t,t+1} A_{t+1} \right\} + M_t + S_t M_t^*.$$

I am assuming that he/she has access to a set of state-contingent securities denominated in units of government currency that span all states of nature. Markets are complete and I denote with A_t the generic time-t payoff of a portfolio of such securities held at time $t-1$; $Q_{t,t+1}$ is the nominal discount factor from time t to time $t+1$. In writing (3), note the term γ multiplying the value of private money held from time $t-1$, expressed in units of government currency. As we already described, the private issuer can decide to destroy a fraction of the money issued. By the law of large numbers, this is seen by the household as a deterministic reduction of its own holdings, although which tokens are still valid remains unknown until the beginning of period t. On the contrary when the sunspot shock hits, $S_t = 0$, the entire stock of private money has no value.[11] Finally, to complete the description of the budget constraint, the household has a constant endowment Y of goods.

[11]Note that $S_T = 0$ for each $T \geq t$ if $S_t = 0$.

He/she is subject to an appropriate borrowing limit of the form

$$\frac{A_t + M_{t-1} + \gamma S_t M^*_{t-1}}{P_t} \geq -E_t \sum_{T=t}^{\infty} R_{t,T} \left(Y + \frac{T_T}{P_T} + \frac{S_T T^*_T}{P_T} \right) > -\infty \qquad (4)$$

for each $t \geq t_0$ and contingency at time t, in which the real stochastic discount factor $R_{t,T}$ between period t and T, with $T > t$, is given by $R_{t,T} \equiv Q_{t,T} P_T / P_t$ with $R_{t,t} \equiv 1$. In (4), the left-hand side of the expression is contingent on whether the private money has burst or not, and on the realization of the exchange rate. The natural borrowing limit (4) is the maximum amount of net debt that the consumer can carry in a certain period of time and repay with certainty, i.e. with current and future net income and assuming that future consumption and asset holdings are going to be equal to zero. The finite borrowing limit is a requirement for consumption to be bounded in the optimization problem.

The consumer chooses the stochastic sequences $\{C_t, A_t, M_t, M^*_t\}_{t=t_0}^{\infty}$ with $C_t, M_t, M^*_t \geq 0$ to maximize (2) under the constraints (3) and (4) for each $t \geq t_0$, given initial conditions A_{t_0},M_{t_0-1}, $M^*_{t_0-1}$.

We now turn to the optimality conditions. The first-order condition with respect to the consumption good is therefore

$$\frac{U_c(C_t)}{P_t} = \lambda_t, \qquad (5)$$

for each $t \geq t_0$ in which λ_t is the Lagrange multiplier attached to the constraint (3). First-order conditions with respect to A_t implies that

$$Q_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \qquad (6)$$

at each time $t \geq t_0$ and contingency at time $t + 1$. The first-order conditions with respect to M_t and M^*_t are:

$$\lambda_t = \frac{1}{P_t} L_m \left(\frac{M_t}{P_t} + \frac{S_t M^*_t}{P_t} \right) + \beta E_t \lambda_{t+1} + \xi_t \qquad (7)$$

and

$$S_t \lambda_t = \frac{S_t}{P_t} L_m \left(\frac{M_t}{P_t} + \frac{S_t M^*_t}{P_t} \right) + \beta \gamma E_t \{S_{t+1} \lambda_{t+1}\} + \xi^*_t, \qquad (8)$$

respectively, at each time $t \geq t_0$, in which ξ_t and ξ^*_t are the non-negative Lagrange multipliers associated with the constraints $M_t \geq 0$ and $M^*_t \geq 0$, respectively. The Kuhn-Tucker conditions are $\xi_t M_t = 0$ and $\xi^*_t M^*_t = 0$.

As shown in the Appendix, the following household's intertemporal budget constraint holds with equality,

$$\mathcal{W}_t + E_t \sum_{T=t}^{\infty} R_{t,T} \left(Y + \frac{T_T}{P_T} + \frac{S_T T_T^*}{P_T} \right) = E_t \sum_{T=t}^{\infty} R_{t,T} \left(C_T + \frac{i_T}{1+i_T} \frac{M_T}{P_T} + \Delta_T \frac{S_T M_T^*}{P_T} \right), \tag{9}$$

in each contingency at time t and for any history looking forward from that contingency. We have defined

$$\mathcal{W}_t \equiv \frac{A_{t-1} + M_{t-1} + \gamma S_t M_{t-1}^*}{P_t}$$

$$\Delta_T \equiv 1 - \gamma E_T \left\{ Q_{T,T+1} \frac{S_{T+1}}{S_T} \right\}.$$

With regard to the two issuers of money, we have already characterized their flow budget constraints. To conclude the presentation of the model, we specify their policy instruments. The government sets the path of interest rates and transfers, i.e. the stochastic sequences $\{i_t, T_t\}_{t=t_0}^{\infty}$ and let the household decides how much to hold of money and bonds. Since $B_t^g = B_t^T + X_t$, the central bank also needs to specify the path of reserves, the sequence $\{X_t\}_{t=t_0}^{\infty}$.[12] The private issuer of currency sets γ and the stochastic sequence $\{T_t^*\}_{t=t_0}^{\infty}$.

2 Equilibrium

Equilibrium in the market for the interest-bearing securities denominated in government currency requires that

$$B_t = B_t^g,$$

in which B_t are household's holdings of government debt. Equilibrium in the cash market for the two currencies implies that supply and demand equalize for each currency

$$M_t = M_t^g,$$

$$M_t^* = M_t^{*p}.$$

The state-contingent private securities are in zero-net supply within the private sector. In any case, I set $A_t = B_t$ at all times and contingency, since the net-asset position of agents is only composed by bonds in equilibrium. Consumption is equal to the constant endowment

$$C_t = Y.$$

[12] See Benigno (2020a,b) and Woodford (2000)..

This implies that the marginal utility of nominal income, λ_t, is $\lambda_t = U_c(Y)/P_t$ and therefore that the nominal and real discount stochastic discount factors are $Q_{t,T} = \beta^{T-t}P_t/P_T$ and $R_{t,T} = \beta^{T-t}$. Without losing generality, I can assume that $U_c(Y) = 1$.

An equilibrium is a set of stochastic processes $\{P_t, S_t, i_t, M_t, M_t^*, B_t, T_t, T_t^*, \xi_t, \xi_t^*\}_{t=t_0}^{\infty}$ with $\{P_t, S_t, i_t, M_t, M_t^*, T_t^*, \xi_t, \xi_t^*\}_{t=t_0}^{\infty}$ non-negative, a non-negative constant variable γ, with $0 \leq \gamma \leq 1$, satisfying

$$\frac{1}{1+i_t} = \beta E_t \left\{ \frac{P_t}{P_{t+1}} \right\} \tag{10}$$

$$\frac{1}{P_t} = \frac{1}{P_t} L_m \left(\frac{M_t}{P_t} + \frac{S_t M_t^*}{P_t} \right) + \beta E_t \left\{ \frac{1}{P_{t+1}} \right\} + \xi_t \tag{11}$$

$$\frac{S_t}{P_t} = \frac{S_t}{P_t} L_m \left(\frac{M_t}{P_t} + \frac{S_t M_t^*}{P_t} \right) + \beta\gamma E_t \left\{ \frac{S_{t+1}}{P_{t+1}} \right\} + \xi_t^* \tag{12}$$

$$\frac{W_t}{P_t} + E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{T_T + S_T T_T^*}{P_T} \right) = E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{i_T}{1+i_T} \frac{M_T}{P_T} + \Delta_T \frac{S_T M_T^*}{P_T} \right), \tag{13}$$

with

$$W_t \equiv M_{t-1} + B_{t-1} + \gamma S_t M_{t-1}^*$$

$$\Delta_t = 1 - \beta\gamma E_t \left\{ \frac{S_{t+1}}{P_{t+1}} \frac{P_t}{S_t} \right\}$$

and

$$M_t^* = T_t^* + \gamma M_{t-1}^*, \tag{14}$$

$$M_t + \frac{B_t}{1+i_t} = T_t + M_{t-1} + B_{t-1}, \tag{15}$$

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{T_T}{P_T} + \frac{S_T T_T^*}{P_T} \right) < \infty, \tag{16}$$

together with the Kuhn-Tucker conditions $\xi_t M_t = 0, \xi_t^* M_t^* = 0$, given initial conditions M_{t_0-1}, B_{t_0-1}, $M_{t_0-1}^*$ and the stochastic process $\{h_t\}_{t=t_0}^{\infty}$. There are four degrees of freedom to specify policy for the two issuers of currency. As discussed earlier, we assume that the government sets the stochastic sequences $\{i_t, T_t, X_t\}_{t=t_0}^{\infty}$ while the private issuer sets γ and $\{T_t^*\}_{t=t_0}^{\infty}$. Note again that $B_t = B_t^T + X_t$.

Equation (10) follows from (6) by taking the conditional expectation at time t and using $\lambda_t = U_c(Y)/P_t$ and $U_c(Y) = 1$. Equations (11) and (12) follow from (7) and (8), respectively, using $\lambda_t = 1/P_t$. Equation (13) follows from (9) using $R_{t,T} = \beta^{T-t}$, together with goods and asset market equilibrium. The boundary condition (16) is derived from (4) having used $R_{t,T} = \beta^{T-t}$.

It is important to note that (13) is an intertemporal resource constraint of the economy as a whole, while there is no intertemporal budget constraint for the two issuers of money that should necessarily hold in equilibrium.

Regarding policy, we make the following assumptions.

Assumption 1 *The government sets a constant interest rate policy $i_t = i$ at each $t \geq t_0$, with $i \geq 0$, and the following transfer policy*

$$\frac{T_t}{P_t} = \frac{i}{1+i}\frac{M_t}{P_t} - (1-\beta)\tau \tag{17}$$

for each $t \geq t_0$ and for some τ positive, and sets a positive level of reserves, $X_t > 0$ for each $t \geq t_0$. The private currency issuer sets $T_t^* = \mu M_{t-1}^*$ and $\gamma = 1$ at each $t \geq t_0$ with $\mu \geq 0$, if it wants to achieve a constant non-negative growth of its money or $T_t^* = 0$ and a γ such that $0 \leq \gamma < 1$, if it wants to achieve a negative growth rate of its money supply.

The above assumption underlines some important differences in the way the two monetary policies are modelled, due to the different nature of the two currencies. Key is that the government can issue cash and interest-bearing securities, which include central bank's reserves, whereas the private issuer supplies only (digital) cash. For this reason, the government can set its policy in terms of the interest rate while the private issuer can only discipline the growth rate of its money supply. Moreover, the treasury can run a real tax policy, which is going to be critical for determining the price level in government currency at least in the single-currency case.[13]

2.1 Equilibria with only government currency

A first result comes by inspection of the equilibrium conditions.

Proposition 1 *There is always an equilibrium in which the private money is worthless and government currency is valued. In this case, the price level is determined at the value*

$$P_{t_0} = \frac{M_{t_0-1} + B_{t_0-1}}{\tau}$$

at time t_0 and $P_{t+1} = P_t\Pi$ for each $t \geq t_0$ in which $\Pi = \beta(1+i)$.

[13] The reader should note that the reason for why the treasury can follow a real tax policy is because the central bank is guaranteeing treasury's solvency at any equilibrium prices, see Benigno (2020a,b).

The first part of the Proposition can be proved by inspecting the equilibrium conditions: note that $S_t = 0$ always satisfies them. In this equilibrium, given the government policy, it follows that (13) implies that

$$\frac{M_{t-1} + B_{t-1}}{P_t} = \tau \tag{18}$$

at all time, which then determines the price level at time t_0 given initial conditions M_{t_0-1} and B_{t_0-1}. The inflation rate is set by (10) given the interest rate policy: $\Pi = \beta(1 + i)$. The supply of money is determined by equation (11) given the path of prices.

This result follows from the asymmetries used to model the two currencies, in line with the differences between existing cryptocurrencies and government currency. The latter can rely on taxation power as opposed to the first and circulate also via debt securities rather than just tokens. As a consequence, the private currency can be worthless if agents believe so.

The reverse result – that there is always an equilibrium with a zero value for government money – does not hold. There are two reasons for this: the first is the trade in interest-bearing securities issued in government currency; the second is the government policy of interest-rate pegging and real taxes. Setting the nominal interest rate fixes the inflation rate, while the real tax policy pins down the price level. In this way, the value of government money is never zero.

To see this result, consider for simplicity the case in which the private currency never bursts and, therefore, the economy is in a perfect foresight equilibrium. Suppose by contradiction that P is infinite and the price level P^* in private currency is finite, with $P^* = P/S$. Therefore, equation (11) is verified.

Note, however, that (13) to (15) imply, in a perfect-foresight equilibrium, the following transversality condition

$$\lim_{T\to\infty} \beta^{T-t} \left\{ \frac{M_{T-1} + B_{T-1}}{P_T} + \frac{\gamma M^*_{T-1}}{P^*_T} \right\} = 0.$$

Since P is infinite, it follows from the above condition that $\lim_{T\to\infty} \beta^{T-t} M^*_{T-1}/P^*_T = 0$, which, used in (13) together with (17), implies that (18) holds and therefore that the price level in government currency is finite.[14] This contradicts the initial claim that the value of government currency is zero.

The equilibrium with only government currency of Proposition 1 is the benchmark to study the challenges that the circulation of another currency could bring about. In this respect, it is worth commenting on the choice of the monetary-fiscal regime of

[14] Note that $\lim_{T\to\infty} \beta^{T-t} M^*_{T-1}/P^*_T = 0$ implies that an intertemporal resource constraint holds, separately, for the private issuer of currency. The proof mimics the derivation of (21) given in the Appendix, but in a perfect-foresight equilibrium.

Assumption 1. It is known in the literature that an interest-rate policy, either active or passive, is not sufficient to determine the path of the price level. Neither a money-supply rule can succeed. As a strategy for comparison, I have built a benchmark equilibrium in which at least there are no problems of determinacy. To this end, I have relied on a real tax policy to determine the price level, as in the fiscal theory of the price level. The objective of the following analysis is to study whether the circulation of another currency can challenge the perfect control of prices that the central bank has in this ideal situation. Alternatively, I could have specified another tax policy implying indeterminacy of prices, which could have blurred the comparison once introducing the private currency.

2.2 Equilibria with competing currencies

I know characterize the conditions under which there are equilibria with competing currencies. In these equilibria $\xi_t = \xi_t^* = 0$.

Consider first the case in which the government set a strictly positive interest rate, combining (10) and (11) we obtain

$$\frac{i}{1+i} = L_m \left(\frac{M_t}{P_t} + \frac{S_t M_t^*}{P_t} \right). \tag{19}$$

Since $i > 0$, the right-hand side of the above expression should remain positive and, therefore $S_t M_t^* / P_t$ should remain finite in any possible history and capped by $\bar{m}$. It follows that

$$\lim_{T \to \infty} E_t \left\{ \beta^{T-t} \frac{S_T M_{T-1}^*}{P_T} \right\} = 0. \tag{20}$$

As shown in the Appendix, by using the above limiting condition and iterating the flow budget constraint (14) forward, I obtain that

$$\gamma \frac{S_t M_{t-1}^*}{P_t} + E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{S_T T_T^*}{P_T} \right) = E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\Delta_T \frac{S_T M_T^*}{P_T} \right), \tag{21}$$

which combined with (13) still implies that (18) holds, using (17), and therefore that the price level is deterministic and grows at the rate Π. Note, indeed, that in this case the intertemporal resource constraint (13) coincides with that of the single-currency economy. It is worth emphasizing the result that the competing currency does not affect the path of prices and inflation rate in government currency, at least when $i > 0$.

To obtain the path of the exchange rate, we can now combine (10) to (12) to obtain

$$S_t = \gamma E_t S_{t+1}.$$

Let us denote with $\tilde{S}_t$ the exchange rate in a history in which the private money still circulates. The above equation can be written as

$$\tilde{S}_{t+1} = \frac{1}{\gamma\alpha}\tilde{S}_t$$

which characterizes the path of the exchange rate between the two currencies conditional on the existence of the private currency. Note, first, the standard result of a constant exchange rate when the probability of the burst is zero, $\alpha = 1$, and there is no destruction of currency, $\gamma = 1$. Otherwise, a positive probability of the burst, a positive destruction rate of private money, all points toward a steady appreciation of the private currency to generate same expected return as government currency. In this way, there is a departure from the standard constant-exchange-rate result of Kareken and Wallace (1981).

I now investigate the conditions on μ for this equilibrium to exist. Suppose first that the private issuer sets $\gamma = 1$ and $M_t^* = (1+\mu)M_{t-1}^*$. Then, for $S_t M_t^*/P_t$ to remain bounded it should be that $(1+\mu) \leq \alpha\Pi$, since prices in government currency grows at the rate Π and the exchange rate appreciates at the rate α. Note that a positive probability of a burst requires a lower growth of the private-money supply for an equilibrium with the two currencies to coexist.

Consider now the case in which $M_t^* = \gamma M_{t-1}^*$. The two money coexists if $1 \leq \alpha\Pi$. Setting $\gamma < 1$ requires the same inequality as if money is constant, since in the former case the exchange rate appreciates to offset the destruction in money supply.

An additional interesting result is that the growth rates of the two moneys can be different in an equilibrium in which both currencies compete for liquidity services. When $1 + \mu < \alpha\Pi$, real money balances in private currency are shrinking over time and converge to zero in the long run, therefore real money balances in government currency are rising over time to reach the limit imposed by (19). This implies that the growth rate of government money is higher than the inflation rate Π and, therefore, higher than that of private money. We have an example of Gresham's law in which the "bad" money, the one with higher growth rate, crowds out the "good" money.

Things are different when $i = 0$, in which case $L_m(\cdot) = 0$ and (13) simplifies to

$$\frac{M_{t-1} + B_{t-1} + \gamma S_t M_{t-1}^*}{P_t} + E_t \sum_{T=t}^{\infty} \beta^{T-t}\left(\frac{S_T T_T^*}{P_T}\right) = \tau,$$

since $\Delta_t = 0$ at all times.[15] In this case, the intertemporal resource constaint of the economy does not collapse to that of the single-currency case, having used (17), because (20) does not hold. The important consequence is that P_t can be now state contingent depending on the exchange rate between the two currencies. Consider

[15] Equation (12) with $\xi_t^* = 0$ and $L_m(\cdot) = 0$ implies $\Delta_t = 0$.

first the case in which $T_t^* = 0$ and $0 < \gamma < 1$. The private money coexists with the government money and the price level is higher in the state in which the private money survives, $h_t = 1$, than when it bursts, $h_t = 0$ and $S_t = 0$. Moreover, in state $h_t = 1$, price and exchange rate are not determined. The indeterminacy of the exchange rate carries over into an indeterminacy of the price level. This result can be intuited by noting that when $L_m(\cdot) = 0$ the private money is a bubble, which is wealth for the household. The key question is to ask how it is possible that it is not ruled out in equilibrium. When $L_m(\cdot) = 0$ and $M_t^* = \gamma M_{t-1}^*$, the first-order condition (12) implies that

$$\frac{S_t M_{t-1}^*}{P_t} = \beta^{T-t} E_t \left\{ \frac{S_T M_{T-1}^*}{P_T} \right\}. \tag{22}$$

Like indeed a bubble, the real value of the private money is exactly equal to its expected discounted value, which is not necessarily zero in equilibrium. Note, indeed, that (13) to (15) imply the following transversality condition

$$\lim_{T \to \infty} \beta^{T-t} E_t \left\{ \frac{M_{T-1} + B_{T-1} + \gamma S_T M_{T-1}^*}{P_T} \right\} = 0.$$

Since $M \geq 0$, a positive discounted value for private money is compensated by the expectation that B will be negative in discounted terms. In this equilibrium, the treasury accumulates risk-free private claims.[16] The same tax policy that was key to determine the price level in the benchmark case is now allowing for the existence of the bubble in private money with the indeterminacy consequences on the price level.[17]

Consider now the case in which the private issuer sets $T_t^* = \mu M_{t-1}^*$ and $\gamma = 1$ therefore

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{S_T T_T^*}{P_T} \right) = \mu E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{S_T M_{T-1}^*}{P_T} \right). \tag{23}$$

Moreover, (12) implies that

$$\frac{S_t M_{t-1}^*}{P_t} = \left(\frac{\beta}{1+\mu} \right)^{T-t} E_t \left\{ \frac{S_T M_{T-1}^*}{P_T} \right\}. \tag{24}$$

Combining (23) and (24), we obtain:

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\frac{S_T T_T^*}{P_T} \right) = \mu \frac{S_t M_{t-1}^*}{P_t} E_t \sum_{T=t}^{\infty} (1+\mu)^{T-t}. \tag{25}$$

[16] This does not bring about any loss of control of the risk-free rate for the central bank, since it can issue a positive amount of reserves X_t and set their interest rate. Note that $B_t = B_t^g = B^T + X$. Therefore, in this equilibrium, it is the treasury position B^T that becomes negative.

[17] Changing the tax rule would not allow the exact comparison with the benchmark single-currency case.

The only way the sum of transfers in (25) can be finite is when $\mu = 0$ in which case transfers are zero. Both moneys coexist, the private money is again a bubble and the price level in government currency becomes indeterminate.

Proposition 2 *In an equilibrium with competing currencies: (i) if the government sets a positive interest rate and the private issuer a non-negative growth rate of money then the growth rate of private money should satisfy* $(1+\mu) \leq \alpha\beta(1+i)$*, if the private issuer destroys money at any rate* γ*, with* $0 < \gamma < 1$*, then* $1 \leq \alpha\beta(1+i)$*; or (ii) if the government sets* $i = 0$ *then* $\mu = 0$ *or any* γ*, with* $0 < \gamma < 1$*.*[18] *In case (i), the path of the price level in government currency is determined, but the exchange rate* S *is not; in case (ii) both are indeterminate.*

Proposition 2 carries two interesting implications. First, when the government sets a positive interest rate policy the second currency causes no problem for the determinacy of the price level and inflation rate. However, when the interest rate is at the zero-lower bound, the price level is indeterminate depending on the value of the private money and its exchange rate. The government loses full control of the price level, because private money circulates as a bubble. When the sunspot shock hits and private money disappears, the wealth loss leads to a fall in the price level.

This second result reinforces the problem of indeterminacy of the exchange rate, as in Kareken and Wallace (1981) and recently restated by Schilling and Uhlig (2019), bringing it at the level of indeterminacy of the price level in government currency.

2.3 Equilibria with only private currency as a medium of exchange

We now study equilibria in which the private money prevails as a medium of exchange on the government currency, conditional on not bursting. For this to be the case, it should be that $\xi_t > 0$ and $\xi_t^* = 0$. To study equilibria, note that we can write (12), on a history in which the private money does not burst and it is the only used as a medium of exchange, as

$$\frac{\tilde{S}_t}{\tilde{P}_t} = \frac{\tilde{S}_t}{\tilde{P}_t} L_m \left(\frac{\tilde{S}_t M_t^*}{\tilde{P}_t} \right) + \alpha\beta\gamma \frac{\tilde{S}_{t+1}}{\tilde{P}_{t+1}}, \tag{26}$$

where a "tilde" variable denotes the variable conditional on that history.

[18] The Proposition extends to the case in which private-money growth can be negative following the rationale of footnote 9. Case (i) would still require $(1+\mu) \leq \alpha\beta(1+i)$, allowing for $\mu < 0$. Case (ii) would require $\mu \leq 0$ in place of $\mu = 0$.

Proposition 3 *In an equilibrium with only private money used as a medium of exchange on histories in which it does no burst: (i) when there is no satiation of liquidity, if the government sets a positive interest rate and the private issuer a non-negative growth rate of money then the growth rate of private money should satisfy* $(1+\mu) < \alpha\beta(1+i)$*. If the private issuer destroys money at any rate* γ*, with* $0 < \gamma < 1$*, then the relevant condition is* $1 < \alpha\beta(1+i)$*; ii) with satiation of liquidity, if the government sets a positive interest rate, the private issuer of money should set* $\mu = 0$ *or destroy money at any rate* γ*, with* $0 < \gamma < 1$*. In case (i), the path of the price level in government currency is determined, but the exchange rate* S *is not; in case (ii) both are indeterminate.*

To prove the Proposition, consider first the case in which the growth rate of private money is non-negative, $M_t^* = (1+\mu)M_{t-1}^*$ with $\mu \geq 0$ and $\gamma = 1$. We can multiply (26) with M_t^* and write it as

$$m_{t+1}^* = \frac{1+\mu}{\alpha\beta}\left(1 - L_m(m_t^*)\right) m_t^*, \tag{27}$$

in which we have defined $m_t^* \equiv \tilde{S}_t M_t^*/P_t$. In what follows, we require that $1 - L_m(m_t^*)$ is positive for all $m_t^* > 0$.[19]

The proof distinguishes three cases: 1) a stationary solution for m_t^*; 2) a decreasing path for m_t^* converging to zero; 3) an increasing path.

I first discuss the existence of equilibria in which $m_t^* < \bar{m}$ and therefore $L_m(\cdot) > 0$ at all points in time along the histories in which private money does not burst and is used as a medium of exchange. Boundedness of m_t^* implies that (20) and (21) hold and therefore that the price level in government currency is deterministic, determined by (18) and growing at the rate Π. Then, the equilibrium condition (11) imposes a lower bound on real money balances since it implies

$$L_m(m_t^*) \leq 1 - \frac{\beta}{\Pi}, \tag{28}$$

and therefore

$$m_t^* \geq L_m^{-1}\left(1 - \frac{\beta}{\Pi}\right).$$

This lower bound is decreasing with the rate of inflation in government currency. The difference equation (27) has a stationary solution $\tilde{m}^*$ implicitly defined by

$$L_m(\tilde{m}^*) = 1 - \frac{\alpha\beta}{1+\mu},$$

[19] This requirement means that as the nominal (implicit) interest rate on private currency becomes very high, the demand for real money balances shrinks to zero.

and in which $L_m(\cdot) > 0$. Considering the bound (28), this stationary solution is an equilibrium provided $(1+\mu) < \alpha\Pi$. The growth rate of private money should be sufficiently low with respect to the inflation rate in government currency, adjusted by the probability α. When $m^*_{t_0} = \tilde{m}^*$ the exchange rate at time t_0 should be at

$$S^*_{t_0} = \frac{\tilde{m}^* P_{t_0}}{M^*_{t_0}} = \tilde{m}^* \frac{B_{t_0-1} + M_{t_0-1}}{\tau(1+\mu)M^*_{t_0-1}},$$

where in the second equality we have used P_{t_0}, which is determined by (18), and $M^*_{t_0} = (1+\mu)M^*_{t_0-1}$. Moreover, to characterize the path of the exchange rate, note that (10) to (12) imply

$$\gamma E_t S_{t+1} > S_t,$$

and therefore

$$\tilde{S}_{t+1} > \frac{\tilde{S}_t}{\gamma\alpha}. \tag{29}$$

The government currency should sufficiently depreciate on a path in which the private money is the only one used, as a medium of exchange.

Second case. The difference equation (27) has also solutions in which m^*_t converges to zero in an infinite period of time. These solutions are ruled out because they violate the bound (28). This result is interesting considering the literature on price determination, since it shows that currency competition can eliminate inflationary equilibria, without requiring implausible assumptions on the demand of real money balances.[20] What is key in our context is not simply competition with any type of currency but with a currency that has always a positive value. Indeed, a finite and positive value for P is critical to determine the bound (28).

The last case to consider is when m_t grows without bounds and the satiation point $L_m(\cdot) = 0$ is reached at some point in time. After that point, (24) holds and the only way the present discounted of transfers can be finite is to have $\mu = 0$, see equation (23) again. Note furthermore that if $i = 0$, and $L_m(\cdot) = 0$, it should necessarily be that also government currency is used in any possible history, i.e. $\xi_t = 0$ in (11). Therefore, to rule out this possibility, I focus on a positive i. When $M^*_t = M^*$, $M_t = 0$ and $L_m(\cdot) = 0$ in the histories in which only private money is used as a medium of exchange, the equilibrium conditions collapse to

$$\frac{1}{1+i} = \beta E_t \left\{ \frac{P_t}{P_{t+1}} \right\}, \tag{30}$$

$$\frac{S_t}{P_t} = \beta E_t \frac{S_{t+1}}{P_{t+1}}, \tag{31}$$

[20] Obstfeld and Rogoff (1983) assume that $\lim_{m^*_t \to 0} L_m(m^*_t)\, m^*_t > 0$, implying that real money balances are bounded below at even very high interest rate levels.

$$\frac{B_t + S_{t+1}M^*}{P_{t+1}} = \tau, \tag{32}$$

together with (29). The private money is again a bubble, as shown in equation (31), whose value is sustained by the taxation policy of the government. In this equilibrium, government prices are state contingent. Equilibrium condition (32) implies that P_{t+1} is higher when $h_{t+1} = 1$, i.e. the value $\tilde{P}_{t+1}$, than when $h_{t+1} = 0$. We can write equation (31) as

$$S_t = \alpha\beta\tilde{S}_{t+1}\frac{P_t}{\tilde{P}_{t+1}} < \alpha\beta\tilde{S}_{t+1}E_t\left\{\frac{P_t}{P_{t+1}}\right\} = \frac{\alpha}{1+i}\tilde{S}_{t+1}.$$

The inequality in the above expression follows by using the result that $\tilde{P}_{t+1}$ is higher than the value taken by P_{t+1} in state $h_{t+1} = 0$. In the last equality, we have used (30). The exchange rate of the private currency appreciates, conditional on not bursting. However, prices and exchange rate are indeterminate and, therefore, the government loses control of the price level.

The proof, along the three cases discussed above, can be adapted to the case in which the private issuer destroys money, i.e. a positive γ less than one. The same difference equation (27) applies, but with $\mu = 0$. Repeating the same steps, the condition for the existence of stationary equilibria is $1 \leq \alpha\Pi$ while equilibria with satiation are always possible provided $i > 0$.

2.4 Summing up

We now summarize the results of the previous sub-sections and comment on the policy implications. Three are the main implications that I will discuss in turn along the following headlines: 1) losing control of the price level; 2) multiple equilibria, 3) Hayek's view.

Losing control of the price level

The first result, drawn from case ii of Proposition 3, says that there is always an equilibrium with satiation of liquidity and with only usage of private tokens as medium of exchange, conditional on the sunspot shock not to realize. In this equilibrium, private tokens are constant in supply, $\mu = 0$, or destroyed at any rate γ and interest rate is positive, $i > 0$. Although private money does not have any non-pecuniary value, it is considered as wealth by the household, whose variation over time affects the price level in government currency. The price level is not determinate and depends on the exchange rate, which is itself indeterminate. When the sunspot shock realizes, the price level drops. Note that this equilibrium holds for any interest rate sets by the government. Only when $i = 0$ government money coexists with private money as a medium of exchange. This result is interesting in light of how some cryptocurrencies are created, like Bitcoins, that are going to reach a constant supply of tokens. This could lead to full satiation of liquidity, no matter what the central bank sets for its policy rate, jeopardizing the control of prices, and ruling out the government currency in the role of medium of exchange. As I have already mentioned, the real tax policy followed by the government, which was critical to determine prices in the single-government-currency case, is now responsible of fueling the bubble. A different tax policy, in which government liabilities are always bounded like in a Ricardian policy, will rule out the private-money bubble, but will lead to indeterminacy of the price level in all other remaining equilibria.

Multiple equilibria

The second important result is that there are multiple equilibria. To see it, set the interest-rate policy at $i > 0$ and a positive growth rate of private money, $\mu > 0$, with $1 + \mu < \alpha\beta(1 + i)$. There are three types of equilibria: one in which only government money is used as a medium of exchange, see Proposition 1, one in which both coexist, see Proposition 2 case i, and another one in which only private money serves as a medium of exchange conditional on no bursting, see Proposition 3 case i. The three equilibria can also be ranked with the one with only private money as medium of exchange to give the highest welfare. Indeed, the marginal utility of real money balances is in that case $1 - \alpha\beta/(1 + \mu)$, as opposed to $1 - 1/(1 + i)$ in the other two equilibria. But, why can't the households coordinate on the best equilibrium? The failure is in the relative price mechanism (exchange rate variations over time) which is the solo incentivizer for households to use one money rather than another. This mechanism fails because it does not convey information on the growth rates of money supplies. The household takes the exchange-rate path as given. To clarify the discussion, let's abstract from the sunspot shock and set $\alpha = 1$.

Given the policies $i > 0$ and $\mu > 0$, with $1 + \mu < \beta(1 + i)$, there are three possible paths of the exchange rate consistent with the same policies: in the first, the exchange rate is zero because private money is worthless, $S_t = 0$; in the second, the exchange rate is constant at $S_t = S$; in the third, the private money appreciates over time $S_{t+1} > S_t$. However, in their portfolio choices, households consider prices as given. If they expect a constant path for the exchange rate between the two currencies, they would hold both moneys irrespective of the lower growth rate of the private one. If they expect $S_{t+1} > S_t$, they would hold only private money at the same growth rate.

Disregarding the equilibria in which $S_t = 0$ that depend on the possibility that an unbacked money can be worthless, the reason for the multiplicity is in the Gresham's law I have discussed when both moneys coexist and $1 + \mu < \beta(1 + i)$. The two moneys can indeed be simultaneously used as a means of exchange even if they have different growth rates. They would have exactly the same return, but the good one, with lower growth rate, is disappearing in real terms providing over time lower and lower purchasing power. Under the same condition $1 + \mu < \beta(1 + i)$, the private money can also dominate in return the government money and be the only used as a means of exchange. The government currency remains a store of value.

Kareken and Wallace (1981) finds a similar Greshman's law, but they do not have the same implications for equilibrium multiplicity. In their framework, money has only the property of store of value. The exchange rate can be either a positive S, in which case the two moneys coexist, $S = 0$ or $S = \infty$ in which only one money exists. They do not have equilibria in which the exchange rate of the private currency appreciates over time.

The multiplicity of equilibria is further exacerbated in our framework when $\mu = 0$ or for any γ in the range $0 < \gamma < 1$. In this case, there are also equilibria in which private money is valued even it it is a pure bubble and the government loses control of prices and medium-of-exchange properties for its currency.

Hayek's view

I am now able to comment on Hayek (1974)'s view that forces of competition should bring the best money – the one with higher return or lower inflation. Hayek's intuition in many of his writings leans on the discovery mechanism that the information provided by prices should bring about. On the contrary, the above analysis has shown that prices do not convey all the relevant information to households in choosing the best money. The correct discovery mechanism should also include knowledge about the rate of growth of private money and on how this would result into a lower inflation rate, if all agents adopt it. A coordination problem arises like in multiple-equilibria context. It should be clear, however, that adding more currencies, and then more competition, does not change results and cannot eliminate the multiplicity of equilibria.

Another important implication is that welfare never decreases with currency competition. To have a possibility to coexist, the unbacked money should be at least as good as the government money. When it is a better money, it can provide a higher return and a lower value for the marginal utility of real money balances. Therefore, looking from a pure welfare perspective, the government should not combat the adoption of new fiat currencies.

Welfare can also be at the first best when the private money is constant in supply or reduced at a certain rate.[21] However, in this case, the government might be worried about losing not only medium-of-exchange properties for its currency but also the control of the price level.

Conversely, when the growth rate of private money is positive, the government retains the control of inflation and prices. In this respect, the path of prices in government currency would be exactly the same independently of the existence or not of the private currency. However, some concern should also arise in this case,

[21] Note that when $\mu = 0$ there are also other equilibria with no satiation of liquidity, as shown in Propositions 2 and 3.

connected to one important assumption we made on the existence of private and public debt in government currency and on taxes levied in that currency. I have not modelled currency competition in those markets, which were critical for the existence of government currency. Risking to lose medium-of-exchange properties could open the way to lose other properties, putting at risk its existence.

Another way to see the results of the three Propositions above is to ask how government should set policy so as to absolutely preclude equilibria in which private currency is used, when $\mu > 0$. The government could fix the interest rate in the range $0 < i < (1+\mu)/(\alpha\beta)-1$, keeping inflation in the interval $\beta < \Pi < (1+\mu)/\alpha$. Interest rates and inflation are then bounded above by the growth rate of private money, adjusted by the probability of the burst. To intuit these bounds, one should observe that in this model currency competition acts in the direction of facilitating certain transactions. To compete and exclude other currencies, the government should offer a better money, one with higher return and lower inflation. Seen from this perspective, the analysis could be in line with Hayek (1974): currency competition can improve the welfare of consumers if the growth rate of private money is sufficiently low and the competing money correspondingly good.

However, government objectives for monetary policy are generally broader than just merely liquidity premia in money markets. Central banks also seek to moderate the fluctuations of the economy. Along these lines, the literature offers justifications for a positive inflation rate, such as the need to avoid the zero lower bound on nominal interest rates or to "grease the wheels" of the economy in the presence of downward nominal wage rigidity.[22] Taking these considerations into account in the simple model of this paper would imply that the bound on inflation can be too stringent for the attainment of other objectives, whenever the growth rate of private money is low. The desired or optimal inflation target could be high enough to enter the region of multiple equilibria where government money might lose its medium of exchange function and be at survival risk.

In the next Section, we investigate whether introducing a central bank digital currency can change these results.

3 Competing with a central bank digital currency

I now extend the analysis to consider a different environment in which central bank's reserves provide also liquidity services to households. We can think at this framework as one in which households are allowed to keep interest-bearing deposits at the central bank in line with recent proposals of central bank digital currency. The other way to introduce central bank digital currency through tokenization will not change the

[22] On the first justification, see Eggertsson and Woodford (2003); on the second, among others, Benigno and Ricci (2011) and Dupraz et al. (2018).

analysis of Section 1 once M_t is understood to capture digital rather than (only) material cash.

To simplify the analysis of this Section, we eliminate uncertainty and therefore the possibility that the private currency can burst. I also simplify the process of private money creation to

$$M_t^{*p} = T_t^* + M_{t-1}^{*p}$$

with $T_t^* \geq 0$, therefore not allowing for destruction in private money, which in previous analysis was having nearly the same implications as in the case in which money growth was zero.

The utility function of the household is now, in a perfect-foresight model,

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ U(C_t) + L\left(\frac{M_t}{P_t} + \frac{B_t}{P_t} + \frac{S_t M_t^*}{P_t}\right) \right\}$$

which is subject to the flow budget constraint

$$\begin{aligned} A_t + B_t + \tilde{B}_t + M_t + S_t M_t^* + P_t C_t \quad \leq \quad & (1+i_{t-1})A_{t-1} + (1+i_{t-1}^g)(B_{t-1} + \tilde{B}_{t-1}) + \\ & + M_{t-1} + S_t M_{t-1}^* + P_t Y + T_t + S_t T_t^*. \end{aligned} \tag{33}$$

The reader should note that I have separated private non-defaulted bonds, A_t, and government bonds and, among the latter ones, those that the households use for liquidity purposes, B_t, and those that are left to be only store of value, $\tilde{B}_t$. It could be argued that it is arbitrary to exclude private non-defaulted bonds from providing liquidity services. This assumption can be justified on the ground that there is a subtle difference between private and government non-defaulted debt securities. Private bonds need to respect a solvency condition for them to be free of default, whereas government bonds are default-free by definition since the central bank is always solvent with respect to its own liabilities. The argument in the function $L(\cdot)$ includes only securities that are default free without the need to satisfy a solvency condition, in their respective currency.

In the constraint (33), the interest rate on government bonds, i_t^g, can be now different from that on private bonds, i_t. The key aspect to underline here is that i_t^g is set by the government (central bank), denoting the interest-rate on reserves, and applies directly to any security issued by the government whether or not this is used for liquidity purposes; i_t^g is not a market rate but a policy choice.

By inspecting the above two constraints, it follows that cash in government currency is now dominated by reserves unless $i_t^g = 0$, therefore the economy will be cashless in equilibrium, but only for what concerns government currency. I set $M_t = 0$ at all times. In any case, the possibility that cash can be used restricts i_t^g to be non-negative, i.e. $i_t^g \geq 0$. The household's problem is subject to an appropriate borrowing limit.

The first-order conditions with respect to C_t, A_t, B_t, $\tilde{B}_t$ and M_t^* are now, respectively, given by

$$\frac{U_c(C_t)}{P_t} = \lambda_t \tag{34}$$

$$\lambda_t = \beta(1+i_t)\lambda_{t+1} \tag{35}$$

$$\lambda_t = \frac{1}{P_t}L_m\left(\frac{B_t}{P_t} + \frac{S_t M_t^*}{P_t}\right) + \beta(1+i_t^g)\lambda_{t+1} + \xi_t \tag{36}$$

$$\lambda_t = \beta(1+i_t^g)\lambda_{t+1} + \tilde{\xi}_t \tag{37}$$

$$S_t\lambda_t = \frac{S_t}{P_t}L_m\left(\frac{B_t}{P_t} + \frac{S_t M_t^*}{P_t}\right) + \beta S_{t+1}\lambda_{t+1} + \xi_t^*, \tag{38}$$

in which I have used the same notation for the Lagrange multipliers as in Section 1 and where $\tilde{\xi}_t$ is the Lagrange multiplier associated to the constraint $\tilde{B}_t \geq 0$. The Kuhn-Tucker conditions are given by $\xi_t B_t = 0$, $\tilde{\xi}_t \tilde{B}_t = 0$, $\xi_t^* M_t^* = 0$ for the non-negative multipliers.

In what follows, I do not repeat all derivations done in Section 1, but I will just remark the important changes. Consider goods market equilibrium, $C_t = Y$, assets market equilibrium, $M_t^* = M_t^{*p}$, $A_t = 0$, $B_t + \tilde{B}_t = B_t^g$ and $B_t^g = B_t^T + X_t$ in which B^T is treasury debt, X_t are central bank reserves, while B_t^g are total government liabilities. Using equilibrium in the goods market and the constant endowment, equation (35) shows that

$$1 + i_t = \frac{1}{\beta}\frac{P_{t+1}}{P_t}. \tag{39}$$

Note, however, that i_t is not the policy rate, so control of inflation in this model comes only indirectly and could be also influenced by the private money supply. Equation (36) combined with (35) implies that

$$\frac{1}{P_t}\frac{i_t - i_t^g}{1+i_t} = \frac{1}{P_t}L_m\left(\frac{B_t}{P_t} + \frac{S_t M_t^*}{P_t}\right) + \xi_t, \tag{40}$$

having used goods market equilibrium, $C_t = Y$ and the assumption $U_c(Y) = 1$. Equation (37) combined with (35), and equation (38) combined with (36) imply, respectively, that

$$\frac{i_t - i_t^g}{1+i_t} = \tilde{\xi}_t, \tag{41}$$

$$\frac{\beta}{P_{t+1}}\left[(1+i_t^g)S_t - S_{t+1}\right] = \xi_t^*. \tag{42}$$

Therefore, $i_t \geq i_t^g \geq 0$ and $S_{t+1} \leq (1+i_t^g)S_t$ in any equilibrium.

As shown in the Appendix, the intertemporal budget constraint of the household, using goods market equilibrium, can be written as

$$\frac{(1+i^g_{t_0-1})B^g_{t_0-1}}{P_{t_0}}+\frac{S_{t_0}M^*_{t_0-1}}{P_{t_0}}+\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(\frac{T_t}{P_t}+\frac{ST^*_t}{P^*_t}\right)=\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(\frac{i_t-i^g_t}{1+i_t}\frac{B^g_t}{P_t}+\Psi_t\frac{S_tM^*_t}{P_t}\right), \tag{43}$$

in which

$$\Psi_t \equiv \left(1-\frac{S_{t+1}}{(1+i_t)S_t}\right),$$

and

$$B_t+\tilde{B}_t=B^g_t \tag{44}$$

$$B^g_t=B^T_t+X_t. \tag{45}$$

Moreover, the budget constraints for the two issuers of currency are now

$$M^*_t=T^*_t+M^*_{t-1}, \tag{46}$$

$$B^g_t=(1+i^g_{t-1})B^g_{t-1}+T_t, \tag{47}$$

and the following boundary condition should hold

$$\sum_{T=t_0}^{\infty}\beta^{T-t_0}\left(\frac{T_T}{P_T}+\frac{S_TT^*_T}{P_T}\right)<\infty. \tag{48}$$

An equilibrium is a set of sequences $\left\{P_t,S_t,i_t,i^g_t,M^*_t,B_t,\tilde{B}_t,X_t,B^T_t,B^g_t,T_t,T^*_t,\xi_t,\xi^*_t,\tilde{\xi}_t\right\}_{t=t_0}^{\infty}$ with $\left\{P_t,S_t,i_t,i^g_t,X_t,B_t,\tilde{B}_t,M^*_t,T^*_t,\xi_t,\xi^*_t,\tilde{\xi}_t\right\}_{t=t_0}^{\infty}$ non-negative, satisfying the equilibrium conditions (39) to (48), together with the Kuhn-Tucker conditions $\xi_tB_t=0$, $\tilde{\xi}_t\tilde{B}_t=0$, $\xi^*_tM^*_t=0$, given initial conditions $B^g_{t_0-1}$, $M^*_{t_0-1}$.[23] There are four degrees of freedom to specify policy for the two issuers of currency. The government sets the sequences $\{i^g_t,T_t,X_t\}_{t=t_0}^{\infty}$ while the private issuer sets $\{T^*_t\}_{t=t_0}^{\infty}$.

The following assumptions define the policies for the two issuers.

Assumption 2 *The government sets a constant interest rate policy $i^g_t=i^g$ at each $t\geq t_0$, with $i^g\geq 0$, and the following transfer policy*

$$\frac{T_t}{P_t}=(i_{t-1}-i^g)\frac{B^g_{t-1}}{P_t}-(1-\beta)\tau \tag{49}$$

for each $t\geq t_0$ and for some τ positive, and set a positive level of central bank's reserves $X_t>0$ at each point in time. The private currency issuer sets $T^*_t=\mu M^*_{t-1}$ at each $t\geq t_0$ with $\mu\geq 0$, to achieve a constant growth rate, $1+\mu$, of its money supply.

[23] Note again that B^T can be negative in which case the government accumulates private assets at the rate i_t on illiquid bonds.

3.1 Equilibrium with one money

I consider first equilibria in which only one money is used as a medium of exchange. Results are striking. As before, there is always an equilibrium in which the unbacked currency is worthless. However, there is never an equilibrium in which only the private money is used as a medium of exchange, unlike Section 2.3.

Proposition 4 *When interest-bearing government liabilities provide liquidity services, they will be always used as medium of exchange.*

The proof is by contradiction. Suppose that $\xi_t > 0$ in (40). Since central bank's reserves are positively supplied at a set interest rate i^g, then it should be that $\tilde{\xi}_t = 0$ in (41). Therefore $i_t = i^g$, which in equation (40) implies a contradiction, i.e. that $\xi_t \leq 0$, proving the Proposition. It also follows that $\tilde{B}_t = 0$ and $B_t^g = B_t$.

In an equilibrium in which $M_t^* = 0$ at all times, using the policies of Assumption 2, equations (39), (40) and (43) imply, respectively, that

$$1 + i_t = \frac{1}{\beta}\Pi_{t+1} \tag{50}$$

$$\frac{i_t - i^g}{1 + i_t} = L_m\left(\frac{B_t}{P_t}\right) \tag{51}$$

$$(1 + i_{t-1})\frac{B_{t-1}}{P_t} = \tau \tag{52}$$

for each $t \geq t_0$. The price level P at time t_0 is determined by (52) given B_{t_0-1}. Use now the flow budget constraint of the government

$$(1 + i_{t-1}^g)\frac{B_{t-1}}{P_t} + \frac{T_t}{P_t} = \frac{B_t}{P_t} \tag{53}$$

and the transfer policy (49) to substitute T_t into (53) to obtain

$$\frac{B_t}{P_t} = \beta\tau.$$

Using this result into (51), it follows that i_t is also constant and determined by the interplay between the tax parameter τ and i^g as

$$1 + i = \frac{1 + i^g}{1 - L_m\left(\beta\tau\right)}.$$

Given the constancy of i, equation (50) implies a constant inflation rate $\Pi = \beta(1+i)$. Note that when taxation is sufficiently high, $\beta\tau \geq \bar{m}$, the first best is achieved. Unlike Section 1, it is the tax policy that determines the supply of liquidity and no longer the interest rate policy.

3.2 Equilibrium with competing currencies

I now discuss equilibria in which both currencies are used as a medium of exchange. In this case, equation (42) implies that $S_{t+1} = (1 + i^g)S_t$, while equation (40) is

$$\frac{i_t - i^g}{1 + i_t} = L_m \left(\frac{B_t}{P_t} + \frac{S_t M_t^*}{P_t} \right). \tag{54}$$

What complicates the analysis, with respect to the simple model of Section 1, is the fact that the interest-rate policy set by the government is now no longer sufficient to determine whether the liquidity constraint is or is not binding. I start from characterizing equilibria in which both moneys coexist and the first best with full satiation is reached. Therefore $i_t = i^g$. In this case, the inflation rate in government currency is determined by the interest-rate on reserves, $\Pi_t = \Pi = \beta(1 + i^g)$. Therefore, $S_{t+1} = S_t \Pi / \beta$. Recall that in equilibrium the present discounted value of private transfers should be finite

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \left(\frac{S_t T_t^*}{P_t^*} \right) = \mu \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{S_t M_{t-1}^*}{P_t} < \infty,$$

which given the law of motions of S, M^* and P is only possible when $\mu = 0$. Let's set, therefore, a constant supply of private money at M^*. In this case, equation (43) can be written as

$$\frac{(1 + i_{t_0-1})B_{t_0-1}}{P_{t_0}} + \frac{S_{t_0} M^*}{P_{t_0}} = \tau,$$

which is not pinning down P_{t_0} separately from S_{t_0}. Using the flow budget constraint of the government, equation (53), together with the tax policy, it also follows that

$$\frac{(1 + i_{t_0-1})B_{t_0-1}}{P_{t_0}} = \frac{B_{t_0}}{P_{t_0}} + (1 - \beta)\tau.$$

Combining the above two equations, we obtain

$$\frac{B_t}{P_t} + \frac{S_t M^*}{P_t} = \beta\tau$$

for each $t \geq t_0$, which is the argument in the right-hand side of (54). An equilibrium in which $i_t = i^g$ and both currencies coexist as medium of exchange requires an appropriate level of taxation, i.e. $\tau \geq \bar{m}/\beta$.

Proposition 5 *When interest-bearing government liabilities provide liquidity services, given the monetary policy regime of Assumption 2, in an equilibrium with full satiation of 1iquidity and currency competition* $\tau \geq \bar{m}/\beta$ *and* $\mu = 0$ *for any* $i^g \geq 0$. *The price level and the exchange rate are not determined.*

There is an important difference with respect to the case in which only cash (or digital cash) provides liquidity services, see Proposition 2, case *ii*. There, the government could control, with its interest-rate policy, the conditions under which the economy is satiated with liquidity. Setting the nominal interest rate to zero achieves the first best. Government and private cash were endogenously adjusting to fulfill liquidity at any equilibrium exchange rate, and the increase in cash was absorbed at the same tax rate by reducing the supply of "illiquid" bond securities. Indeed, the parameter τ determined the supply of cash in the two currencies and of illiquid government bonds. Here, the government can still achieve the first best but not anymore with the interest-rate policy. It needs to raise enough taxes and the parameter τ now determines just the supply of liquid securities, which includes the "liquid" government bonds. Private-money growth should be set at $\mu = 0$ as in Proposition 2 and price level and exchange rate are indeterminate, as well.

I will now analyze equilibria in which both moneys coexist but where liquidity is not satiated, at least to start with. Recall the equilibrium condition:

$$\frac{i_t - i^g}{1 + i_t} = L_m \left(\frac{B_t}{P_t} + \frac{S_t M_t^*}{P_t} \right). \tag{55}$$

When $i_t > i^g$, the right-hand side is positive. Since $B_t > 0$, then $0 \leq S_t M_t^*/P_t^* \leq \bar{m}$ at all times. Given that real money balances in private money are bounded, an intertemporal budget constraint holds for the private issuer, which used into (43) implies that (52) holds for each $t \geq t_0$.

Using the budget constraint of the government, equation (53), we can still obtain that

$$\frac{B_t}{P_t} = \beta\tau.$$

Since $L_m(\cdot) > 0$, τ should be below $\bar{m}/\beta$ in this equilibrium. Equation (38) can be written as

$$m_{t+1}^* = \frac{\beta}{1 + \mu} [1 - L_m (\beta\tau + m_t^*)] m_t^*, \tag{56}$$

in which $m_t^* = S_t M_t^*/P_t$.

There are three cases to consider: a stationary solution for m_t^*, increasing and decreasing paths. First case. There is an equilibrium in which m_t^* is constant at the level defined by

$$L_m (\beta\tau + m^*) = 1 - \frac{\beta}{1 + \mu},$$

which together with (55) implies that $(1 + i) = [(1 + i^g)(1 + \mu)]/\beta$. The interest rate on illiquid bonds depends on the interest rate on government currency and the growth rate of private money. The inflation rate is given by $\Pi = (1 + i_g)(1 + \mu)$, which is also affected by the growth rate of private money. For this equilibrium to exist, the exchange rate should be

$$S_{t_0}^* = \frac{m^* P_{t_0}}{M_{t_0}^*} = m^* \frac{(1 + i_{t_0-1}) B_{t_0-1}}{\tau(1+\mu) M_{t_0-1}^*},$$

at time t_0, where in the second equality we have used P_{t_0}, which is determined by (52), and $M_{t_0}^* = (1+\mu) M_{t_0-1}^*$.

For values of $S_{t_0} < S_{t_0}^*$, there are also equilibria in which m_t^* decreases to zero in an infinite period of time. In this case, the equilibrium converges in the limit to the one with only a government currency, discussed in Section 3.1. But, in all the transition, prices and inflation depend also on the growth rate of private money.

The third case to consider is when $S_{t_0} > S_{t_0}^*$. These solutions are ruled out as equilibria. Indeed in a finite period of time the satiation level $\bar{m}$ will be reached, and from that point in time we know, given the analysis of Proposition 5, that $\mu = 0$ and $\tau \geq \bar{m}/\beta$. Here, instead, it is assumed that $\tau < \bar{m}/\beta$, otherwise there will be full satiation of liquidity.

Proposition 6 *In the model in which interest-bearing government liabilities provide liquidity services, given the monetary policy regime of Assumption 2 , in equilibria in which liquidity is not satiated and both moneys coexist as a medium of exchange* $\tau < \bar{m}/\beta$. *The price level in government currency is determined but the exchange rate is not, whose level characterizes the type of equilibrium. The inflation rate and interest rate on illiquid bonds depends on the growth rate of private money.*

I can draw some further conclusions comparing the results of this Section with those of Section 2.4. In both cases there are multiple equilibria. However, when interest-bearing government liabilities provide liquidity services there cannot be equilibria in which only private money is held as a medium of exchange. This result comes at some costs. When taxation is low, $\tau < \bar{m}/\beta$, there can be equilibria with competing currencies no matter what is the rate of growth of private money. In all these equilibria, the government does not have anymore full control of the inflation rate in its currency, unlike Section 2.4, which now depends on the policy followed by the private issuer of currency and on the random realization of the exchange rate. In any case, however, welfare with currency competition is never below the single-currency case, since the marginal utility of real money balances is never above that with only government currency.

To regain full control of the inflation rate, the government should set a relatively high level of taxation $\tau \geq \bar{m}/\beta$ to satiate liquidity. In this case, the private money can only coexist with government money if its rate of growth is zero. Although the government has full control of the inflation rate with its policy rate, the price level is now indeterminate.

4 Conclusion

This paper analyzes models of coexistence between government and privately-issued currencies both in an environment in which government or private cash compete as a medium of exchange and in a context in which interest-bearing government securities compete with privately-issued (digital) cash. Depending on the environment restrictions might arise on government policies and on the possible equilibria. In any case, welfare is never decreased by currency competition.

I have kept the analysis as simple as possible in order to focus on this important topic, which has recently received a good deal of attention. In particular, private and government currencies are assumed to be perfect substitutes, delivering the same liquidity services. This assumption can be motivated by the characteristics of digital currencies, which can be carried everywhere in electronic wallets, stored on cellular phones that facilitate their use for transaction purposes. Once more and more sellers will accept them as means of payment, the usage of government currency might become less essential and their substitutability increases. Taking into account, instead, imperfect substitutability will make each currency essential for liquidity purposes and their coexistence will require milder relationships between growth rates and interest rate in the two currencies, without altering the results significantly, see Benigno et al. (2022). Most interesting would be to devise a model in which the acceptance or non-acceptance of currencies is endogenous not only for the medium of exchange function but also for other properties of money – a task I leave to future research. Another interesting avenue would be extension to a multi-country world, as a way of studying competition among international reserve currencies and national currencies. Benigno et al. (2022) investigate the consequence for the international monetary system of trading a global currency, finding restrictions on cross-country interest rates and on the exchange rate. Their analysis, which, in any case, does not analyze competition in a closed-economy model, is only focused on the implications that can be derived from asset-price restrictions and does not characterize equilibria on the basis of the policies followed by the issuers of currencies, unlike in this paper.

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5 Appendix

In this Appendix, we derive some key equilibrium conditions of the model.

5.1 Derivation of equation (9)

Consider the flow budget constraint

$$W_t \leq A_t + M_{t-1} + \gamma S_t M_{t-1}^* - P_t C_t + P_t Y + T_t + S_t T_t^*,$$

with

$$W_t = E_t \left\{ Q_{t,t+1} A_{t+1} \right\} + M_t + S_t M_t^*,$$

and divide them P_t to obtain

$$\frac{W_t}{P_t} \leq \frac{A_t}{P_t} + \frac{M_{t-1}}{P_t} + \gamma \frac{S_t M_{t-1}^*}{P_t} - C_t + Y + \frac{T_t}{P_t} + \frac{S_t T_t^*}{P_t}.$$

Define

$$\mathcal{W}_t \equiv \frac{A_t}{P_t} + \frac{M_{t-1}}{P_t} + \gamma \frac{S_t M_{t-1}^*}{P_t}$$

and note that

$$\begin{aligned} \frac{E_t \left\{ Q_{t,t+1} A_{t+1} \right\} + M_t + S_t M_t^*}{P_t} &= E_t \left\{ Q_{t,t+1} \frac{P_{t+1} \mathcal{W}_{t+1} - M_t - \gamma S_{t+1} M_t^*}{P_t} \right\} + \\ & \quad \frac{M_t + S_t M_t^*}{P_t} \\ &= E_t \left\{ R_{t,t+1} \mathcal{W}_{t+1} \right\} + \frac{i_t}{1+i_t} \frac{M_t}{P_t} + \Delta_t \frac{S_t M_t^*}{P_t} \end{aligned}$$

since $R_{t,t+1} = Q_{t,t+1} P_{t+1} / P_t$, $E_t Q_{t,t+1} = 1/(1+i_t)$ and

$$\Delta_t = 1 - \gamma E_t \left\{ Q_{t,t+1} \frac{S_{t+1}}{S_t} \right\}.$$

We can then write the flow budget constraint as

$$E_t \left\{ R_{t,t+1} \mathcal{W}_{t+1} \right\} + \frac{i_t}{1+i_t} \frac{M_t}{P_t} + \Delta_t \frac{S_t M_t^*}{P_t} = \mathcal{W}_t - C_t + Y + \frac{T_t}{P_t} + \frac{S_t T_t^*}{P_t}$$

which iterated forward using the transversality condition

$$\lim_{T \to \infty} E_t \left\{ R_{t,T} \mathcal{W}_T \right\} = 0$$

implies the equilibrium condition (9).

5.2 Derivation of equation (21)

Consider the flow budget constraint of the private issuer of money

$$\frac{S_t M_t^*}{P_t} = \gamma \frac{S_t M_{t-1}^*}{P_t} + \frac{S_t T_t^*}{P_t} \tag{57}$$

which can be written as

$$\frac{S_t M_t^*}{P_t} - \gamma E_t \left\{ R_{t,t+1} \frac{S_{t+1} M_t^*}{P_{t+1}} \right\} + \gamma E_t \left\{ R_{t,t+1} \frac{S_{t+1} M_t^*}{P_{t+1}} \right\} = \gamma \frac{S_t M_{t-1}^*}{P_t} + \frac{S_t T_t^*}{P_t}$$

and therefore

$$\frac{S_t M_t^*}{P_t} - \gamma E_t \left\{ Q_{t,t+1} \frac{S_{t+1}}{S_t} \right\} \frac{S_t M_t^*}{P_t} + \gamma E_t \left\{ R_{t,t+1} \frac{S_{t+1} M_t^*}{P_{t+1}} \right\} = \gamma \frac{S_t M_{t-1}^*}{P_t} + \frac{S_t T_t^*}{P_t}$$

$$\gamma E_t \left\{ R_{t,t+1} \frac{S_{t+1} M_t^*}{P_{t+1}} \right\} + \Delta_t \frac{S_t M_t^*}{P_t} = \gamma \frac{S_t M_{t-1}^*}{P_t} + \frac{S_t T_t^*}{P_t}.$$

Equation (21) is obtained from equation (57) if

$$\gamma \lim_{T \to \infty} E_t \left\{ R_{t,T} \frac{S_T M_{T-1}^*}{P_T} \right\} = 0$$

holds.

5.3 Derivation of equation (43)

Start from

$$\begin{aligned} A_t + B_t^g + S_t M_t^* + P_t C_t \quad \leq \quad & (1 + i_{t-1}) A_{t-1} + S_t M_{t-1}^* + (1 + i_{t-1}^g) B_{t-1}^g + \\ & + P_t Y + T_t + S_t T_t^*. \end{aligned} \tag{58}$$

Define

$$\mathcal{W}_t = \frac{(1 + i_{t-1}) A_{t-1} + S_t M_{t-1}^* + (1 + i_{t-1}^g) B_{t-1}^g}{P_t}$$

and therefore

$$A_t = \frac{P_{t+1} \mathcal{W}_{t+1} - S_{t+1} M_t^* - (1 + i_t^g) B_t^g}{1 + i_t}$$

which can be substituted into the budget constraint to obtain

$$R_{t,t+1} \mathcal{W}_{t+1} + \frac{i_t - i_t^g}{1 + i_t} \frac{B_t^g}{P_t} + \Delta_t \frac{S_t M_t^*}{P_t} + C_t = \mathcal{W}_t + Y + \frac{T_t}{P_t} + \frac{S_t T_t^*}{P_t},$$

with

$$\Delta_t = 1 - \frac{S_{t+1}}{S_t(1+i_t)}$$

and

$$R_{t,t+1} = \frac{P_{t+1}}{P_t}\frac{1}{1+i_t}.$$

Iterating the above equation forward using the transversality condition

$$\lim_{T\to\infty} R_{t,T}\mathcal{W}_T = 0$$

we obtain

$$\sum_{T=t}^{\infty} R_{t,T}\left(C_T + \frac{i_T - i_T^g}{1+i_T}\frac{B_T^g}{P_T} + \Delta_T\frac{S_T M_T^*}{P_T}\right) = \mathcal{W}_t + \sum_{T=t}^{\infty} R_{t,T}\left(Y + \frac{T_t}{P_t} + \frac{S_t T_t^*}{P_t}\right).$$

Using goods market equilibrium and setting $A_t = 0$, we obtain equation (43).