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All-Pay Auctions with Reserve Price and Bid Cap

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Abstract

I study the joint effects of the reserve price and the bid cap in the all-pay auction. I show that in equilibrium bidding, there are: (i) atoms at 0, the reserve price, and the bid cap; and (ii) continuous bidding above the reserve. If the valuations are high enough, the range for continuous bidding vanishes entirely. Further, I characterize environments with multiple equilibria and derive settings with the following features: with two players, the player with the lower valuation can have a positive rent; with three players, active competition for a single prize is possible with totally asymmetric valuations; in both cases, the value of the positive payoff for the "winner" can vary across different equilibria.

1 Introduction

Competitive environments often feature contestants exerting costly efforts in pursuit of winning one of a limited number of valuable prizes. In sports leagues, teams hire athletes, coaches, and managers to compete for championship trophies and prize money; in political campaigns, parties use funds to promote candidates in competition for representation seats; in R&D, firms conduct costly research to compete for patents.

All-pay auctions commonly appear in models of such competitive environments, especially when the effort costs are fully or almost non-recoverable. For instance, Hillman

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and Riley (1989), Baye et al. (1993), and Che and Gale (1998) use all-pay auctions to model and analyze lobbying and campaign expenditures, while Moldovanu and Sela (2003) and Che and Gale (2003) model patent and R&D races.

Reserve prices, as well as expenditure caps, are standard tools in economic design, which often appear in real-life contest settings.[1] For instance, the NBA features both a salary cap for teams as well as a minimum team salary.[2] Similarly, there are salary limits and franchise fees in the NFL, resembling the bid cap and the reserve price, respectively.[3] Some economic environments inherently feature one of the studied attributes (the reserve price *or* the bid cap). For those settings, studying the introduction of the *other* attribute can allow for a greater flexibility, crucial for the contest designer, or the regulator. For instance, sunk costs are typical in R&D.[4] An organizer of an R&D race might try to limit the duplicating expenses by means of an expenditure cap. Another relevant environment is the evaluation of agents' performance by a lenient reviewer: Letina et al. (2020) show that an all-pay auction with a bid cap is optimal in this setting. One reasonable robustness check consists of the analysis of how the equilibrium and the effort profile in that setting change with a fixed effort cost.

While the reserve and the cap in the all-pay auction setting have been studied separately (see Che and Gale (1998), Siegel (2014), Szech (2015)), their *joint* influence on players' behavior has not yet received enough attention. Their joint study is, however, important. On the one hand, their combination allows the designer greater flexibility in the pursuit of the objective.[5] On the other hand, the reserve's and the cap's aggregate effect is not guaranteed to be just the sum of the two. Moreover, and as will be shown is the case, multiple equilibria might emerge limiting the robustness when the two tools are combined.

In this paper, I study the complete information all-pay auction with a reserve price and a bid cap in the presence of two and three players. I completely characterize the equilibria in this environment. I establish the settings in which the same player can be

[1]Depending on the context, the reserve price can take a form of an entry fee, a fixed cost, or something similar.

[2]See, for instance, www.cbssports.com/nba/news/nba-wants-2-5-billion-fee-for-possible-expansion-teams-expects-offers-in-vegas-and-seattle-per-report/ and https://cbabreakdown.com/salary-cap-overview.

[3]www.cnbc.com/2018/10/05/nfl-owners-teams-football.html.

[4]Sunk costs in R&D are studied by Manez et al. (2009).

[5]There are various goals that the principal might desire, from maximizing useful efforts/minimizing wasteful expenditures to selective efficiency/leveling the playground.

either a “winner” or a “looser” in different equilibria (see Corollary 1); settings where a “winner” can have different payoffs across equilibria (see Proposition 4.iii); settings where three players with totally asymmetric valuations can bid actively in equilibrium (see Proposition 4.iv); and settings where the player with a non-top valuation is the “winner” (as illustrated in examples for Propositions 4.iii and 4.iv).

My results, therefore, complement and contrast the previous findings in the literature. In the standard complete information all-pay auction with 1 prize and N players, Baye et al. (1996) show that the only player, who can be the “winner” is the one who has a strictly greater valuation for the prize than the others. Besides, more than two players can actively bid only in non-generic cases. Moreover, for more general all-pay contests, Siegel (2009) shows that under regularity conditions, there are pre-determined sets of “winners” and “losers”, such that “winners” get their rents, same across equilibria, and “losers” get zero.

Besides the immediate relevance for the contest theory, the results in this paper also have implications for the literature on information design. In Muratov (2021), I characterize a mapping between an all-pay auction with the reserve and the cap, and the competing multi-sender information design game, as in Boleslavsky and Cotton (2018). Hence, deriving equilibria in the setting of the current paper also allows for characterizing equilibria in related information design games.

2 Model: Two Bidders Case

Consider an all-pay auction with one prize and two bidders, 1 and 2, whose valuations for the prize are V_1 and V_2, respectively. They submit bids, b_1 and b_2, and always have to pay them. The player, who bids and pays the most, wins the prize.

Suppose that in addition to standard assumptions, there is a reserve price, r, and a bid cap, κ, such that $0 < r < \kappa$. Normalize $\kappa = 1$. A player who bids below the reserve price can never win the item, even if his bid turns out to be the largest; any bid above 1 is discarded. In case of a tie, let bidder 1 obtain the prize with probability ρ_1, and bidder 2 - with probability ρ_2, such that $\rho_1 + \rho_2 = 1$, $\rho_1 \in (0, 1)$.

Let the players be risk-neutral expected utility maximizers, with utility functions

$u_1()$ and $u_2()$, respectively. For a pair of bids (b_1, b_2) the payoff of player i is

$$u_i(b_1, b_2) = \begin{cases} V_i - b_i, & \text{if } b_i > b_k, b_i \in [r, 1] \\ \rho_i \times V_i - b_i, & \text{if } b_i = b_k, b_i \in [r, 1] \\ -b_i, & \text{else.} \end{cases}$$

Let us first state the result of how the equilibrium bidding looks in general. The result is valid for the cases of the two players and the three players, which is considered in section 3.

Lemma 1. *The bids are distributed in the subset of* $[0, 1]$*. The subset has the following properties:*

(i) There is no mass of bids in $(0, r)$

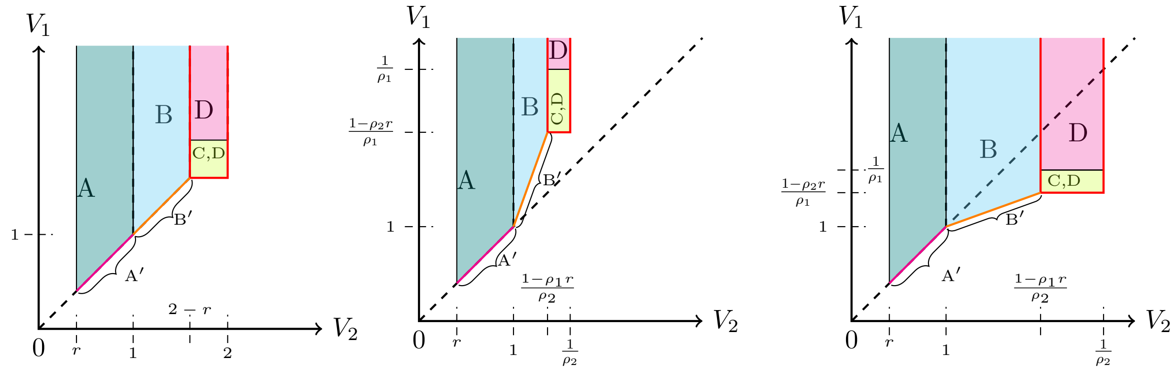
(ii) There can be atoms at $\{0\}$, $\{r\}$, *and* $\{1\}$*. There can be at most one player placing an atom at* $\{r\}$*. There can be no other atoms*

(iii) If there is mass of bids in a set $B \subseteq (r, 1)$*, then* B *is an interval:* $B = (r, \overline{b})$, $r < \overline{b} \leqslant 1$*. At least two players bid everywhere in* $(r, \overline{b})$*. If a third player also bids in the interior of* B*, he distributes it continuously in an interval* $(\underline{b}, \overline{b})$*, where* $\underline{b} \in (r, \overline{b})$*, but is not uniquely determined*

The proof of this lemma is in the appendix A.1.

As for the exact behavior of two players, three qualitatively different types of bidding emerge in equilibria, depending on how high the valuations are. Following the regions in figure 1, when one of the valuations is low, (region A), there can be no atoms besides the ones at 0 and r, with the remaining mass distributed continuously in a subset of $(r, 1)$; when one of the valuations is medium, (region B), there are atoms at 0, r, and 1, as well as continuously distributed mass in the subset of $(r, 1)$; when both valuations are high (regions C and D), there are atoms only. We now list the results for a strict subset of $(V_1, V_2) \in [r, +\infty)^2$, namely the union of the regions A, A′, B, B′, C, and D, as depicted in figure 1. The remaining cases are symmetric to those we describe.[6]

[6]Note also, that there are trivial equilibria for a subset of parameters: if both valuations are below the reserve price, both agents will bid 0 with certainty; if only one valuation is above the reserve price, the player with that valuation will bid r with certainty; and if both valuations are high enough, i.e., if $(V_1, V_2) > \left(\frac{1}{\rho_1}, \frac{1}{\rho_2}\right)$, both players will bid 1 with certainty.



$\rho_1 = 1/2$ $\rho_1 = 3/8$ $\rho_1 = 5/8$

Figure 1: Equilibrium Regions

Proposition 1.i. *If $V_2 < 1$ and $V_2 \leqslant V_1$ (region A), in the unique equilibrium, the players' bids are distributed according to the CDFs:*

$$G_1(b) = \begin{cases} 0, & \text{if } b < r \\ \frac{b}{V_2}, & \text{if } b \in [r, V_2) \\ 1, & \text{if } b \geqslant V_2; \end{cases} \qquad G_2(b) = \begin{cases} \frac{r+V_1-V_2}{V_1}, & \text{if } b < r \\ \frac{b+V_1-V_2}{V_1}, & \text{if } b \in [r, V_2) \\ 1, & \text{if } b \geqslant V_2. \end{cases}$$

Player 1 has a positive equilibrium payoff $U^\star = V_1 - V_2$; Player 2 has a zero equilibrium payoff.

Proposition 1.ii. *If $V_2 = V_1 \leqslant 1$ (region A'), there are multiple equilibria. In every equilibrium the bids are distributed according to the CDFs:*

$$G_1(b) = \begin{cases} \frac{r-t_1}{V_2}, & \text{if } b < r \\ \frac{b}{V_2}, & \text{if } b \in [r, V_2) \\ 1, & \text{if } b \geqslant V_2; \end{cases} \qquad G_2(b) = \begin{cases} \frac{r-t_2}{V_2}, & \text{if } b < r \\ \frac{b}{V_2}, & \text{if } b \in [r, V_2) \\ 1, & \text{if } b \geqslant V_2; \end{cases}$$

where t_i is a parameter that stands for the size of the atom that player i has at r, such that $t_i \in [0, r]$, $t_1 \times t_2 = 0$. Both players have equilibrium payoffs of zero.

If player 1 has a strictly higher valuation, the equilibrium is unique, and only player 1 has an atom at r. If valuations are equal, there are multiple equilibria. Two sources lead to multiplicity: the identity of the player, who has an atom at r; and the size of that atom.

The characterization above is exhaustive, as shown in the Appendix B.1.

Besides the multiplicity of equilibria under equal valuations, the significant departure from the standard setting of Baye et al. (1996) is that there can be no mass of bids in $(0, r)$ in equilibrium. Atoms at 0 and r compensate for that gap. Figure 2 shows the plots of CDFs and comparative statics with respect to the increase in the V_2.

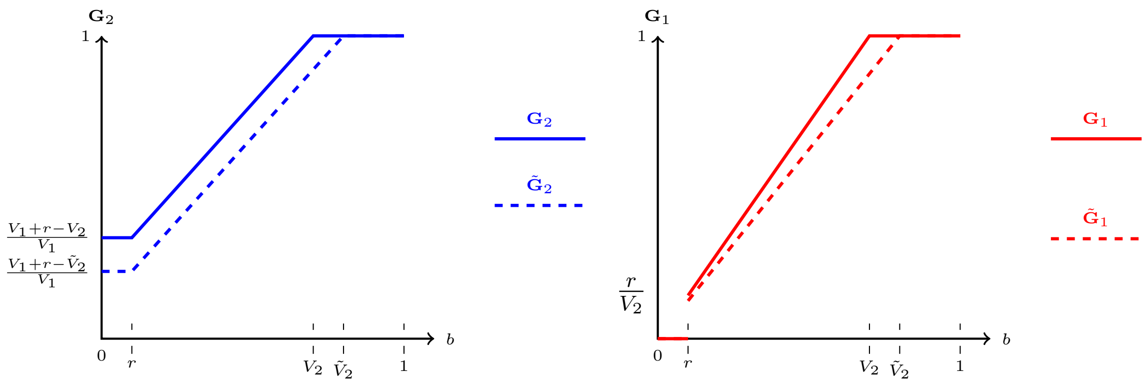


Figure 2: Distributions of bids and comparative statics with respect to V_2

Note that in regions A and A′, the support of bids expands with V_2. As V_2 becomes greater than 1, the support cannot expand further. At that point, equilibria of regions B and B′ apply:

Proposition 2.i. *If* $1 < V_2 \leqslant \frac{1-\rho_1 r}{\rho_2}$, $\rho_2^2 V_2 - \rho_2 < \rho_1^2 V_1 - \rho_1$ *(region B), in the unique equilibrium, the players' bids are distributed according to the CDFs:*

$$G_1(b) = \begin{cases} 0, & \text{if } b < r \\ \frac{b}{V_2}, & \text{if } b \in \left[r, \frac{1}{\rho_1} - \frac{\rho_2}{\rho_1} V_2\right) \\ \frac{1-\rho_2 V_2}{\rho_1 V_2}, & \text{if } b \in \left[\frac{1}{\rho_1} - \frac{\rho_2}{\rho_1} V_2, 1\right) \\ 1, & \text{if } b \geqslant 1; \end{cases} \qquad G_2(b) = \begin{cases} \frac{r+U^\star}{V_1}, & \text{if } b < r \\ \frac{b+U^\star}{V_1}, & \text{if } b \in \left[r, \frac{1}{\rho_1} - \frac{\rho_2}{\rho_1} V_2\right) \\ \frac{1-\rho_2 V_2+\rho_1 U^\star}{\rho_1 V_1} & \text{if } b \in \left[\frac{1}{\rho_1} - \frac{\rho_2}{\rho_1} V_2, 1\right) \\ 1, & \text{if } b \geqslant 1, \end{cases}$$

where $U^\star = \frac{\rho_2 - \rho_1 + \rho_1^2 V_1 - \rho_2^2 V_2}{\rho_1^2} > 0$ *is the equilibrium payoff of player 1. Player 2 has a zero payoff.*

Proposition 2.ii. *If* $1 < V_2 \leqslant \frac{1-\rho_1 r}{\rho_2}$, $\rho_2^2 V_2 - \rho_2 = \rho_1^2 V_1 - \rho_1$ *(region B′), there are multiple* *equilibria. In every equilibrium the players' bids are distributed according to the CDFs:*

$$G_1(b) = \begin{cases} \frac{r-t_1}{V_2}, & \text{if } b < r \\ \frac{b}{V_2}, & \text{if } b \in \left[r, \frac{1}{\rho_1} - \frac{\rho_2}{\rho_1}V_2\right) \\ \frac{1-\rho_2 V_2}{\rho_1 V_2}, & \text{if } b \in \left[\frac{1}{\rho_1} - \frac{\rho_2}{\rho_1}V_2, 1\right) \\ 1, & \text{if } b \geqslant 1; \end{cases} \qquad G_2(b) = \begin{cases} \frac{r-t_2}{V_1}, & \text{if } b < r \\ \frac{b}{V_1}, & \text{if } b \in \left[r, \frac{1}{\rho_1} - \frac{\rho_2}{\rho_1}V_2\right) \\ \frac{1-\rho_2 V_2}{\rho_1 V_1} & \text{if } b \in \left[\frac{1}{\rho_1} - \frac{\rho_2}{\rho_1}V_2, 1\right) \\ 1, & \text{if } b \geqslant 1, \end{cases}$$

where $t_i \in [0, r]$, $t_1 \times t_2 = 0$. Both players have zero payoffs.

It is shown in the appendix B.2 that the above characterization is exhaustive.

There are several changes, as compared to 1.i-1.ii: both players have atoms at 1; the support of continuous bidding shrinks in V_2; player 1's payoff is not equal to $V_1 - V_2$ unless $\rho_1 = \frac{1}{2}$. So, favoritism is effective under the bid cap and high enough valuations: if $\rho_1 > \rho_2$, player 1 can have the payoff that is higher than the standard payoff of $V_1 - V_2$. Player 2's payoff, however, remains equal to 0.

What remains the same is the density of bidding in the continuous part. The upper bound of the continuous part of the support, $\bar{b}$, the sizes of atoms, and the equilibrium payoff of player 1 are determined by the indifference of players between bidding $\bar{b}$ and bidding 1.

Figure 3 shows the CDFs for a typical pair of valuations in region B and the tie-breaking rule $\rho_1 = \rho_2 = 1/2$, as well as the comparative statics with respect to the increase in V_2.

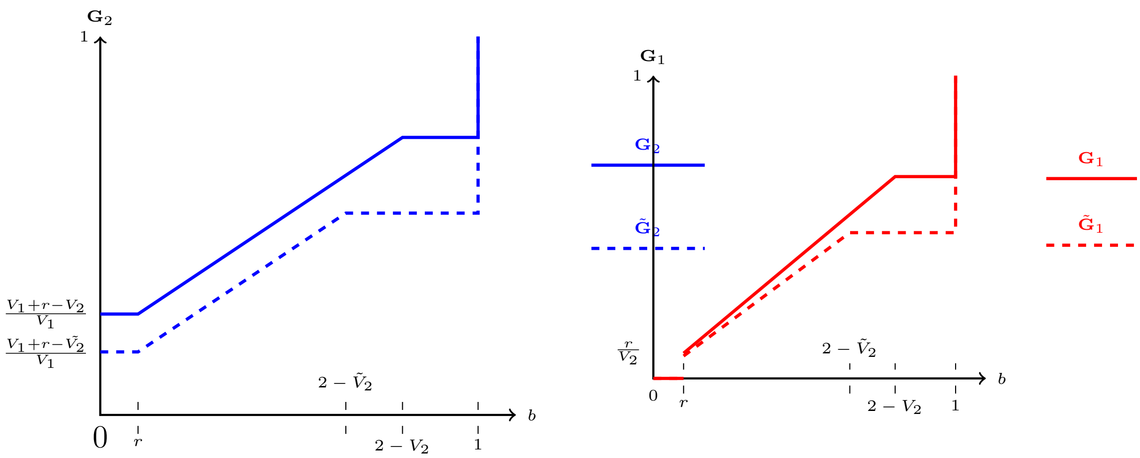


Figure 3: Distributions of bids

Finally, when V_2 is high enough so that $\bar{b}$ decreases all the way to the reserve price, any continuous bidding region shrinks entirely. In regions C and D, in equilibria, the bidding is completely characterized by atoms.

Proposition 3.i. *If* $(V_1, V_2) \in \left[\frac{1-r\rho_2}{\rho_1}, \frac{1}{\rho_1}\right] \times \left[\frac{1-r\rho_1}{\rho_2}, \frac{1}{\rho_2}\right]$*, (region C), there is an equilibrium in which the players bid according to*

$$b_1 = \begin{cases} 0, & \textit{with probability } \frac{1-\rho_2 V_2}{\rho_1 V_2} \\ 1, & \textit{with probability } \frac{V_2 - 1}{\rho_1 V_2} \end{cases} \qquad b_2 = \begin{cases} 0, & \textit{with probability } \frac{1-\rho_1 V_1}{\rho_2 V_1} \\ 1, & \textit{with probability } \frac{V_1 - 1}{\rho_2 V_1} \end{cases};$$

and both players have zero payoffs.

Proposition 3.ii. *If* $(V_1, V_2) \in \left\{\frac{1-\rho_2 r}{\rho_1}\right\} \times \left[\frac{1-r\rho_1}{\rho_2}, \frac{1}{\rho_2}\right]$*, there is a class of equilibra in which the players bid according to*

$$b_1 = \begin{cases} 0, & \textit{with probability } (1-t) \times \frac{1-\rho_2 V_2}{\rho_1 V_2} \\ r, & \textit{with probability } t \times \frac{1-\rho_2 V_2}{\rho_1 V_2} \\ 1, & \textit{with probability } \frac{V_2 - 1}{\rho_1 V_2} \end{cases} \qquad b_2 = \begin{cases} 0, & \textit{with probability } \frac{1-\rho_1 V_1}{\rho_2 V_1} \\ 1, & \textit{with probability } \frac{V_1 - 1}{\rho_2 V_1} \end{cases};$$

where $t \in [0,1]$ *is a free parameter, and both players have zero payoffs.*

Proposition 3.iii. *If* $(V_1, V_2) \in \left[\frac{1-\rho_2 r}{\rho_1}, \infty\right) \times \left[\frac{1-\rho_1 r}{\rho_2}, \frac{1}{\rho_2}\right]$*, (region D), there is an equilibrium, in which the players bid according to*

$$b_1 = \begin{cases} r, & \textit{with probability } \frac{1-\rho_2 V_2}{\rho_1 V_2} \\ 1, & \textit{with probability } \frac{V_2 - 1}{\rho_1 V_2} \end{cases} \qquad b_2 = \begin{cases} 0, & \textit{with probability } \frac{V_1\rho_1 - (1-r)}{V_1 \rho_1} \\ 1, & \textit{with probability } \frac{1-r}{\rho_1 V_1} \end{cases};$$

and player 1 has a positive equilibrium payoff $u_1^\star = V_1 - \frac{1-\rho_2 r}{\rho_1}$

The above characterization in propositions 3.i-3.iii is exhaustive because, under such high valuations and the bidding space limited by 1 from above, there can be no mass in the interior of $(r, 1)$.

Let us reconsider the setting of 3.iii, but switch the role of the players. It follows then that for region $(V_1, V_2) \in \left[\frac{1-\rho_2 r}{\rho_1}, \frac{1}{\rho_1}\right] \times \left[\frac{1-\rho_1 r}{\rho_2}, \infty\right)$, there is an equilibrium with player one mixing between bidding 0 and 1, bringing him a zero payoff, and player two mixing between bidding r and 1, bringing him a positive payoff of $V_2 - \frac{1-\rho_1 r}{\rho_2}$. This observation, combined with the results of propositions 3.i-3.iii allows to state the following:

Corollary 1. *In the non-empty region* $(V_1, V_2) \in \left[\frac{1-\rho_2 r}{\rho_1}, \frac{1}{\rho_1}\right] \times \left[\frac{1-\rho_1 r}{\rho_2}, \frac{1}{\rho_2}\right]$*, there are three qualitatively different equilibria. In this region, each player can be a "winner" (has a positive payoff) or a "loser" (has a zero payoff), or both players an be "losers".*

Corollary 1 shows why it is essential to study the joint influence of the reserve and the cap. Their aggregate effect allows for the switch of the identity of the "winner" while holding the fundamentals fixed. Moreover, the initially disadvantaged player (the one with the lower $\rho_i^2 V_i - \rho_i$) can become the "winner." Such equilibrium behavior might be useful for the principal-designer, who wishes to favor the disadvantaged.

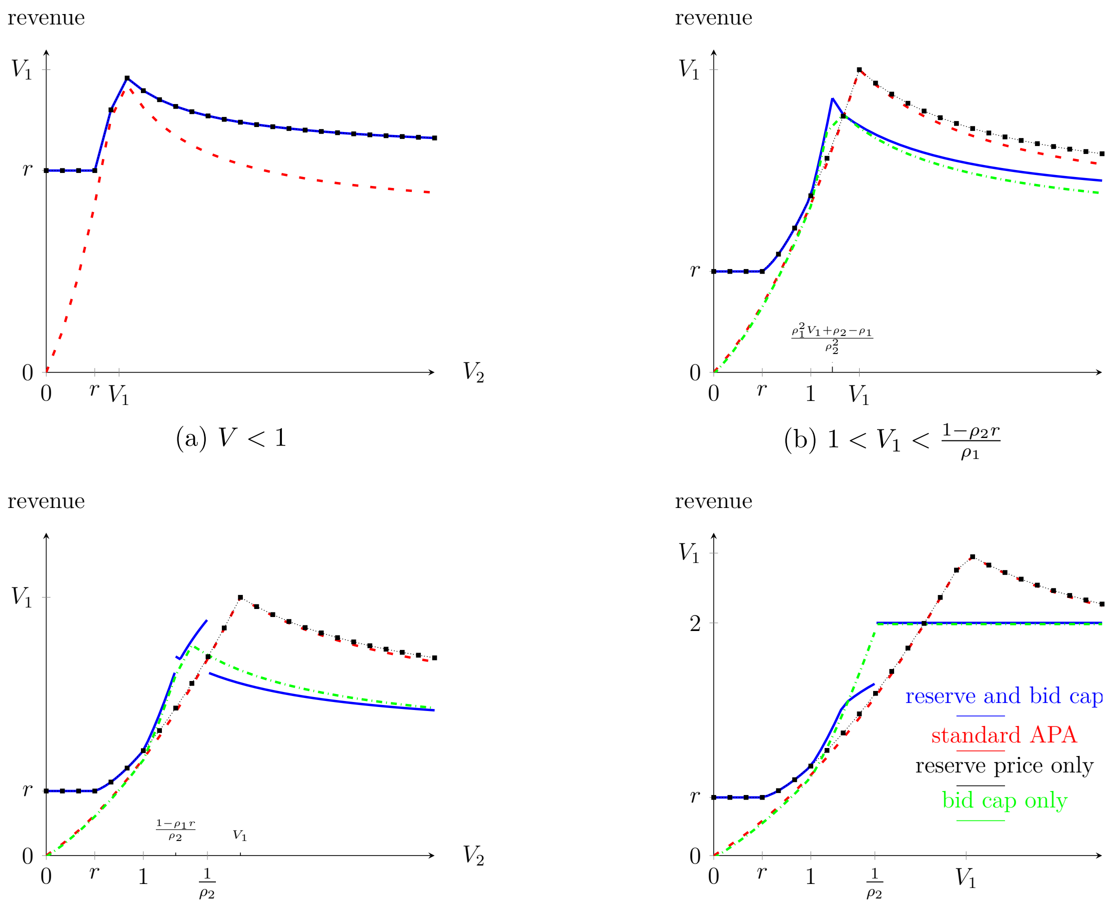
This multiplicity of equilibria that allows for switching of roles and benefiting the disadvantaged happens when both valuations are high enough. In this region of valuations, whoever commits to bidding r with a positive probability, becomes the "winner." And in that region, with high valuations, each player can be the one to commit to bidding r.

2.1 Characterizing expenditures

Knowing the equilibrium distributions of bids allows to compute various expected characteristics of bids, such as, the individual expenditures of each player, the sum of expenditures, the variance of expenditures, and so on. Depending on the exact economic context, and the interpretation of the bids (productive effort in preparation for an exam, a sports team budget, R&D expenditure) the contest organizer might have different objectives - maximize the total average expenditures/efforts; make sure that individual expenditures are above a certain threshold; compress the discrepancy between individual expenditures; increase the probability one specific player wins the prize; and so on . In the appendix C, I provide the expressions for individual average expenditures and their sum, as examples of the target variables, and the ways they behave in the current environment.

In a working paper, Muratov (2021), I show that there is a mapping between all-pay auctions and a class of information design games, in which competing entrepreneurs-senders try to persuade a single receiver to invest into their project, rather than the opponent's, as the one studied in Boleslavsky and Cotton (2018). Average expenses in the all-pay auctions environment correspond to prior expected qualities in the information design environment.

Using the expenditures formulae from that appendix, I plot the sum of the bidders' expenditures as a function of V_2 (figure 4). Several values for V_1 are considered: low ($r < V < 1$), average-low ($1 < V_1 < \frac{1-\rho_2 r}{\rho_1}$), average-high ($\frac{1-\rho_2 r}{\rho_1} < V_1 < \frac{1}{\rho_1}$), and high ($V_1 > \frac{1}{\rho_1}$).[7] Together with the expenditures in the all-pay auction with the reserve price r and the bid cap of 1, the graphs also depict the expenditures in cases of no restrictions; with just the reserve price r; and just the bid cap of 1. It can be seen although there is no uniform ranking, the all-pay auction *with* the reserve price *and* the bid cap can provide the largest total expenditures for some ranges of valuations.



(a) $V < 1$ (b) $1 < V_1 < \frac{1-\rho_2 r}{\rho_1}$ (c) $\frac{1-\rho_2 r}{\rho_1} < V_1 < \frac{1}{\rho_1}$ (d) $V_1 > \frac{1}{\rho_1}$

Figure 4: Revenue comparison for different values of V_1

[7] For parameters with multiple equilibria, the equilibria with maximal expenditures were chosen. Also note that discontinuities in the expenditures are due to multiplicity of equilibria on the boundaries of regions.

3 Three Bidders Case

Studying the case of two bidders in the all-pay auction in the presence of the reserve price and the bid cap has brought new insights, if we compare the results to the ones in the settings of Baye et al. (1996) and Che and Gale (1998). Thus, for a range of parameters, $(V_1, V_2) \in \mathrm{C} \cap \mathrm{D}$, in equilibrium bidding CDFs are characterized by atoms only, but not purely at the top. In the same range of parameters there are three types equilibria. In one, both players bid 0 and 1 only, in another, player 1 bids r instead of 0 and has a positive payoff, and in the third type it is player 2 who bids r instead of 0 and has a positive payoff. Thus, in this region, it is possible that the player with initial disadvantage, i.e. lower valuation of the prize, in equilibrium gets a positive payoff, unlike in the setting of Baye et al. (1996), and that phenomenon is not driven by the tie-breaking rule.

Studying this environment with the third bidder allows to perform a robustness check of the results, derived in the two players case; and it also brings new insights, not possible under the two bidders. Thus, multiple equilibria are possible even for ranges of parameters that result in equilibria with some continuous bidding. The nature of this multiplicity is of a different kind than the one in Baye et al. (1996) with three or more players: in addition to the degree of freedom in when one of the three players joins the continuous bidding, there is multiplicity of the positive equilibrium rent, and the equilibrium support, which all affects the shapes of CDFs. Another new finding is that it is possible for all three players to bid actively for a range of completely asymmetric valuations $V_1 \neq V_2 \neq V_3$, with two players' bidding involving continuous support, and the third player bidding at the top and the bottom.

Throughout this section only the symmetric tie-breaking rule will be considered, i.e 1/2-1/2 chance of getting the item if the two bidders tie, and 1/3-each chance of getting the item if the three bidders tie.

Let us first identify the restrictions on the valuations of player 3, V_3, such that equilibria from section 2 hold; i.e. player 3 stays inactive and effectively only players 1 and 2 are the bidders.

Proposition 4.i. *There are non-empty regions of parameters, such that only players 1 and 2 are active bidders. Depending on how high V_1 and V_2 are, the results are:*

- *For the parameters* $V_1 \geqslant V_2 \geqslant V_3$, $V_2 \in [r, 1)$, *there are equilibria as in the environments of propositions 1.i and 1.ii. Moreover, if* $V_2 > V_3$, *these are all the equilibria*

- *For the parameters* $V_1 \geqslant V_2 \in [1, 2-r)$, $V_3 \leqslant \frac{3V_1V_2}{4+3V_1-5V_2+V_2^2}$, *there are equilibria as in the environments of propositions 2.i and 2.ii, under* $\rho_1 = \rho_2 = 1/2$.

- *For the parameters* $(V_1, V_2) \in [2-r, 2]^2$, $V_3 \leqslant \frac{3V_1V_2}{4+V_1(V_2-1)-V_2}$, *there are equilibria as in the environments of propositions 3.i and 3.ii.*

- *For the parameters* $(V_1, V_2) \in [2-r, \infty) \times [2-r, 2]$, $V_3 \leqslant \frac{3V_1V_2}{3V_1+V_2+r(4-V_2)-4}$, *there is equilibrium as in the environment of proposition 3.iii.*

If $V_3 < \min\{1, V_1, V_2\}$, *these are all the equilibria.*

The values for V_3 in the regions of the equilibria regimes in the above proposition follow from postulating the player 1's and 2's behavior to be as in the results of the section 2, and making it unprofitable for the player 3 to bid anywhere above 0.

Note that when the valuation of the third player is strictly below the bid cap, $V_3 < 1$, and strictly smaller than the other valuations, the analysis is essentially the same as with the two bidders: the third bidder is always inactive.

The case of $V_1 \geqslant V_2 = V_3 = v$ with $v \in [r, 1]$ leads to the multiplicity of equilibria: two bidders, 1 and 2 are always active in (r, v), while the third player joins the bidding in $(\tilde{b}, v)$, $\tilde{b} \in [r, v]$.[8] The indeterminacy of $\tilde{b}$ is what creates the multiplicity of equilibria.[9]

Proposition 4.ii. *Suppose* $V_1 \geqslant V_2 = V_3 = v$, *and* $v \in [r, 1)$. *Then, for each* $\tilde{b} \in [r, v]$ *there is an equilibrium where players bid according to the CDFs:*

$$- \ G_1(b) = \begin{cases} 0, & \text{if } b < r \\ \frac{b}{v} \times \sqrt{\frac{V_1}{V_1+\tilde{b}-v}}, & \text{if } b \in [r, \tilde{b}) \\ \frac{b}{v} \times \sqrt{\frac{V_1}{V_1+b-v}}, & \text{if } b \in [\tilde{b}, v] \end{cases}, \quad - \ G_2(b) = \begin{cases} \frac{r+V_1-v}{\sqrt{V_1(\tilde{b}+V_1-v)}}, & \text{if } b \in [0, r) \\ \frac{b+V_1-v}{\sqrt{V_1(\tilde{b}+V_1-v)}}, & \text{if } b \in [r, \tilde{b}) \\ \frac{\sqrt{b+V_1-v}}{\sqrt{V_1}}, & \text{if } b \in [\tilde{b}, v] \end{cases},$$

$$- \ G_3(b) = \begin{cases} \frac{\sqrt{\tilde{b}+V_1-v}}{\sqrt{V_1}}, & \text{if } b \in [0, \tilde{b}) \\ \frac{\sqrt{b+V_1-v}}{\sqrt{V_1}}, & \text{if } b \in [\tilde{b}, v] \end{cases}.$$

[8] Players 2 and 3 can switch their roles.

[9] Such multiplicity is of equilibria is of the same nature as the one in Baye et al. (1996).

If $V_1 = V_2$, player 1 can have an atom at 0, with the total mass of 0 and r being $\frac{r}{v} \times \sqrt{\frac{V_1}{V_1+\tilde{b}-v}}$. These CDFs describe all equilibria for such set of parameters.

Overall, in this case the higher value bidder has an atom at the reserve prise, while the other two bidders have atoms at the 0. When two players are active in the interior, their bids are uniformly distributed, and when three players are active the bids are distributed in such way, that the maximum of any two bids is distributed uniformly.

The proof for this case is located in D.2.

For the case of the second- and third-order valuations still being equal to each other and taking medium values, the distributions of bids are similar to the case of results on Proposition 4.ii, but now there appears and atom at the bid cap, 1. There is no explicit way to express the equilibrium objects, namely the bidder one's equilibrium payoff and the upper bound of the continuous part of the support. They follow from the system of non-linear equations.

Proposition 4.iii. *For the set of valuations $V_1 \gtreqless V_2 = V_3 = v$, $V_1 \geqslant 1$, $v \in [1,3]$, there is a non-empty subset, such that equilibrium CDFs are*

$$
\begin{aligned}
G_1(b) &= \frac{b}{v} \times \sqrt{\frac{V_1}{\tilde{b}+U}} \mathbb{I}_{\{b \in [r,\tilde{b})\}} + \frac{b}{v} \times \sqrt{\frac{V_1}{b+U}} \mathbb{I}_{\{b \in [\tilde{b},\hat{b})\}} + \frac{\hat{b}}{v} \times \sqrt{\frac{V_1}{\hat{b}+U}} \mathbb{I}_{\{b \in [\hat{b},1)\}} + \mathbb{I}_{\{b \geqslant 1\}} \\
G_2(b) &= \frac{r+U}{\sqrt{V_1(\tilde{b}+U)}} \mathbb{I}_{\{b \in [0,r)\}} + \frac{b+U}{\sqrt{V_1(\tilde{b}+U)}} \mathbb{I}_{\{b \in [r,\tilde{b})\}} + G_3(b) \mathbb{I}_{\{b \geqslant \tilde{b}\}} \\
G_3(b) &= \frac{\sqrt{\tilde{b}+U}}{\sqrt{V_1}} \mathbb{I}_{\{b \in [0,\tilde{b})\}} + \frac{\sqrt{b+U}}{\sqrt{V_1}} \mathbb{I}_{\{b \in [\tilde{b},\hat{b})\}} + \frac{\sqrt{\hat{b}+U}}{\sqrt{V_1}} \mathbb{I}_{\{b \in [\hat{b},1)\}} + \mathbb{I}_{\{b \geqslant 1\}}
\end{aligned}
$$

where U, player 1's positive equilibrium rent, and $\hat{b}$ follow from the system of equations:

$$
\begin{cases}
\frac{1}{3} + \frac{1}{3}\sqrt{\frac{U+\hat{b}}{V_1}} + \frac{1}{3}\frac{U+\hat{b}}{V_1} & = \frac{1+U}{V_1} \\
\frac{1}{3}\sqrt{\frac{U+\hat{b}}{V_1}} + \frac{1}{6}\frac{\hat{b}}{v} + \frac{1}{6}\frac{U+\hat{b}}{V_1} + \frac{1}{3}\frac{\hat{b}}{v}\sqrt{\frac{U+\hat{b}}{V_1}} & = \frac{1}{v}\sqrt{\frac{U+\hat{b}}{V_1}}.
\end{cases}
\tag{A}
$$

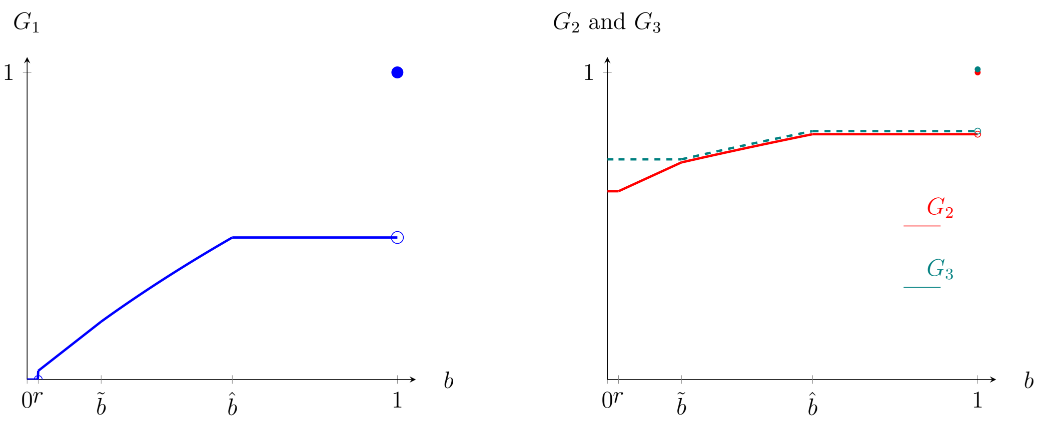
Moreover, for a non-empty subset of valuations, two solutions, $(U, \hat{b})$, to the above system, constitute equilibrium objects. If $U = 0$, player 1 can have an atom at 0, with the total mass of 0 and r being $\frac{r}{v}\sqrt{\frac{V_1}{\tilde{b}}}$.

In the appendix D.3 I derive the algebraic expressions for the regions of valuations, such that the type of equilibrium from the Proposition 4.iii holds. Here I demonstrate the results with several numerical examples.

Let $r = 0.03$, $V_1 = 2.56$, $V_2 = V_3 = 1.5$. Then, the solution to system A that satisfies the equilibrium requirements $U \geqslant 0$, $\hat{b} \geqslant r$ is, approximately,

$$U = 1.07954, \hat{b} = 0.5541.$$

Note that since one of the players 2 or 3 can join the bidding at any $\tilde{b} \in [r, \hat{b}]$, which causes the multiplicity of equilibria. We can plot equilibrium CDFs for this type of equilibria for some choice of $\tilde{b}$.



(a) G_1 (b) G_2, G_3

Figure 5: Distributions of bids in the environment of 4.iii

Consider now different valuations of players 2 and 3 $V_2 = V_3 = 2.57$, fixing $r = 0.03$ and $V_1 = 2.56$, as before. The solutions to system A that satisfy the equilibrium conditions are, approximately:

$$\begin{cases} U^1 & = 0.103775, \hat{b}^1 = 0.04028 \\ U^2 & = 0.009232, \hat{b}^2 = 0.05451. \end{cases}$$

Notice that this example demonstrates that under some parameters a slightly disadvantaged player 1 can have a positive equilibrium rent, and moreover, some parameters can lead to two classes of equilibria.[10]

Comparative statics of equilibrium U and $\hat{b}$ with respect to v is not trivial due to

[10]There are multiple equilibria within each class, because player 3 can join the continuous bidding anywhere in $(r, \hat{b})$, and that also affects the behavior of players 1 and 2. Each class is represented by a single solution $(U, \hat{b})$ to the system A.

non-linearity of the condition of system A and the fact that there can be two pairs $(U, \hat{b})$ that satisfy the equilibrium condition. On the figure 6, we plot the graphs of U and $\hat{b}$ with respect to the value of v, given $V_1 = 2.5$, $r = 0.01$; and compare them we the behavior of U and $\hat{b}$ for the case of players 1 and 2 being active (i.e. the case of $V_3 = 0$). Note the multiplicity of equilibria for v high enough.

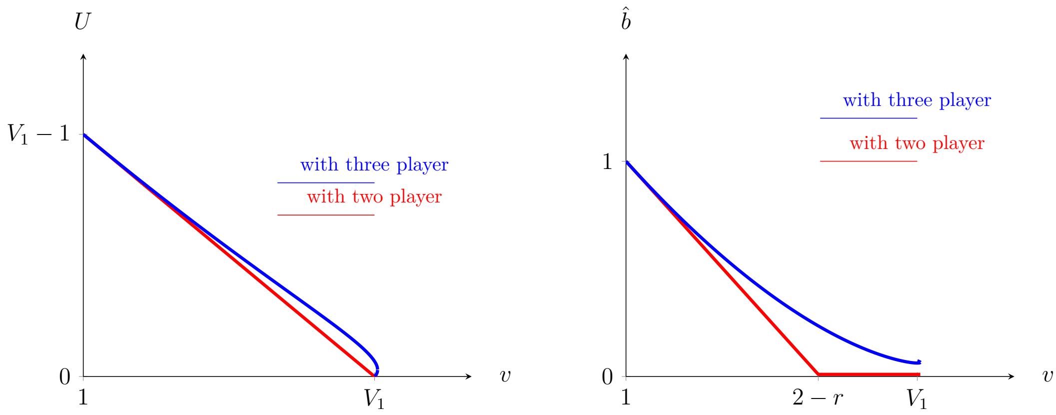


Figure 6: **Comparative statics of U and $\hat{b}$ in 4.iii**

Proposition 4.iv. *There exists a non-empty subset of $(V_1, V_2, V_3) \in (1, +\infty)$ such that there is an equilibrium, where player 1 (2) has an atom at r (0), bids continuously in $(r, \hat{b})$, $\hat{b} > r$, and has an atom at 1, while player 3 only bids 0 and 1 with non-zero probabilities. Player 1 has a non-negative equilibrium payoff U. The equilibrium objects $(U, \hat{b}, P_3)$, where P_3 is the probability of player 3 bidding 0, follow from the system*

$$\begin{cases} \frac{1}{3}V_1 + \frac{1}{6}P_3V_1 + \frac{1}{6}\frac{\hat{b}+U}{P_3} + \frac{1}{3}(\hat{b}+U) & = 1 + U \\ \frac{1}{3}V_2 + \frac{1}{6}P_3V_2 + \frac{1}{6}\frac{\hat{b}}{P_3} + \frac{1}{3}\hat{b} & = 1 \\ \frac{1}{3}V_3 + \frac{1}{6}\frac{\hat{b}+U}{V_1P_3} + \frac{1}{6}\frac{\hat{b}}{V_2P_3} + \frac{1}{3}\frac{(\hat{b}+U)\hat{b}}{V_1V_2P_3^2} & = 1 \end{cases} \tag{B}$$

If $U = 0$, player 1 can have an atom at 0, arbitrarily splitting the mass between 0 and r.

In the appendix D.4 I describe how to establish the region of parameters, in which this type of equilibrium holds. Here I provide examples of parameters for this type of equilibria to demonstrate that such region is non-empty.

For example, for the parameters $V_1 = 2.7$, $V_2 = 2$, $V_3 = 2.66$, $r = 0.07$, the solution to system B, that satisfies equilibrium conditions, is (approximately) $U = 0.729137$, $\hat{b} = 0.165146$, $P_3 = 0.72$. Note that for this solution, the values of CDFs at $\hat{b}$ are less than one, $G_1(\hat{b}) = \frac{\hat{b}}{V_2P_3} = 0.114652$, $G_2(\hat{b}) = \frac{\hat{b}+U}{V_1P_3} = 0.4599$; and that player 3 does not have incentives to bid anywhere in in $(r, \hat{b})$:

$$V_3G_1(b)G_2(b) - b = V_3 \times \frac{b}{V_2P_3} \times \frac{b+U}{V_1P_3} - b$$
$$= b\left(V_3\frac{1}{V_2P_3} \times \frac{b+U}{V_1P_3} - 1\right) \leqslant b\left(V_3\frac{1}{V_2P_3} \times \frac{\hat{b}+U}{V_1P_3} - 1\right) = b \times (-0.150715).$$

Another thing to notice about this example is that player 1's payoff is larger than $V_1 - V_2$ and $V_1 - V_3$. So, the standard payoff result, that under some assumptions in equilibrium the top player's payoff is $V_1 - \max_{i \neq 1} V_i$ does not hold in this environment for this type of equilibrium.

Consider another example of parameters and the correspondent solutions. For $V_1 = 2.62$, $V_2 = 2.621$, $V_3 = 2.875$, $r = 0.01$, there is a couple of solutions that satisfy equilibrium conditions: $U^I = 0.12$, $\hat{b}^I = 0.017578$, $P_3^I = 0.249$; and $U^{II} = 0.00075$, $\hat{b}^{II} = 0.0425$, $P_3^{II} = 0.1454$. Beside the multiplicity of equilibria for these parameters, it is also noteworthy that player 1, the one with positive rent, has the lowest valuation in this example.

Let us now describe two types of equilibria, where all players are active and bids' distributions are atomic only. The first of such two types is the one, where every players' equilibrium payoff is zero, and they all bid 0 and 1:

Proposition 5.i. *There exists a non-empty subset of* $(V_1, V_2, V_3) \in (1, 4)^3$ *such that there is an equilibrium, where players bid* 0 *and* 1, *only. Player* i *bids* 0 *with a probability* P_i, *as in*

$$\begin{cases} P_1 &= \frac{\sqrt{3V_1(V_2-4)(V_3-4)}}{2\sqrt{(4-V_1)V_2V_3}} - \frac{1}{2} \\ P_2 &= \frac{\sqrt{3V_2(V_1-4)(V_3-4)}}{2\sqrt{(4-V_2)V_1V_3}} - \frac{1}{2} \\ P_3 &= \frac{\sqrt{3V_3(V_1-4)(V_2-4)}}{2\sqrt{(4-V_3)V_1V_2}} - \frac{1}{2}. \end{cases} \tag{C}$$

The region is defined by the conditions $(\cap_i\{0 \leqslant P_i \leqslant 1\}) \cap (\cap_i\{V_i \prod_{j \neq i} P_j \leqslant r\})$.

The probabilities in the above proposition follow from solving the system

$$V_i\left(\mathbb{P}\{\text{bid}_j=0\}\mathbb{P}\{\text{bid}_k=0\}+\frac{1}{2}(1-\mathbb{P}\{\text{bid}_j=0\})\mathbb{P}\{\text{bid}_k=0\}+\right.$$
$$\left.+\frac{1}{2}(1-\mathbb{P}\{\text{bid}_j=0\})\mathbb{P}\{\text{bid}_k=0\}+\frac{1}{3}(1-\mathbb{P}\{\text{bid}_j=0\})(1-\mathbb{P}\{\text{bid}_k=0\})\right)-1=0,$$
$$(i,j,k)\in\{1,2,3\}, i\neq k\neq j,$$

with respect to $\mathbb{P}\{\text{bid}_i=0\}\doteq P_i$. There are two solutions to that system, but only one allows for joint possibility of $0\leqslant P_i\leqslant 1$. The restrictions, defining the region, follow from feasibility of probabilities and absence of profitable deviation to bidding $b\in(0,1)$.

Another type of equilibrium with atomic distributions is the one, where one player has a positive equilibrium payoff and bids r instead of 0:

Proposition 5.ii. *There exists a non-empty subset of* $(V_1,V_2,V_3)\in(1,4)^3$ *such that there is an equilibrium, where player 1 bids* r *and* 1, *with probabilities* P_1 *and* $1-P_1$, *and has a payoff* $U\geqslant$; *and players 2,3 bid* 0 *and* 1, *only. Player* $i\in\{2,3\}$ *bids* 0 *with a probability* P_i, *as in*

$$\begin{cases} P_1 &= \frac{\sqrt{3V_1(V_2-4)(V_3-4)}}{2\sqrt{(4-V_1)V_2V_3}}-\frac{1}{2}\\ P_2 &= \frac{\sqrt{3V_2(V_1-4)(V_3-4)}}{2\sqrt{(4-V_2)V_1V_3}}-\frac{1}{2}\\ P_3 &= \frac{\sqrt{3V_3(V_1-4)(V_2-4)}}{2\sqrt{(4-V_3)V_1V_2}}-\frac{1}{2}. \end{cases} \tag{F}$$

Player 1's payoff is

$$U=3-r-\frac{\sqrt{3(4-V_1)V_1}(V_2(V_3-2)-2V_3)}{2\sqrt{(4-V_2)V_2(4-V_3)V_3}}-\frac{V_1}{2}.$$

The region is defined by the conditions $(\cap_i\{0\leqslant P_i\leqslant 1\})\cap(\cap_{i\in\{2,3\}}\{V_i\prod_{j\neq i}P_j\leqslant r\})\cap\{U\geqslant 0\}$. *If* $U=0$, *player 1 is indifferent between* r *and* 0, *and can split the mass between these two bids.*

Note that after we were to consider all permutations of valuations in all of the above propositions of the section 3, we would have characterized all types of equilibria possible under three players, except for those where all three players are inactive; and those that result in players bidding 1 with certainty.

4 Conclusion

In this paper the equilibria of the all-pay auctions with the reserve price and the bid cap were characterized. It was shown that equilibrium bidding is characterized by atoms at 0, the reserve price, and the cap, as well as continuous bidding between the reserve and the cap. It was shown that multiple equilibria are possible for some parameter ranges. In case of three players, the following interesting features can emerge in equilibria: the player with a slight disadvantage can have the positive payoff; three players with completely different valuations can all actively compete for the single prize; the "winning" player can have different payoffs across different equilibria.

A Appendix

A.1 General Characterization of Equilibria

Let us show that all the equilibria in All-Pay Auctions with a reserve price and a bid cap are follow the characterization in lemma 1. We do so in the series of lemma below.

Lemma A.1. *There is no mass of bids above 1*

Proof. Suppose by way of contradiction that there is at least one player, who places a bid greater than 1 with a non-zero probability. Recall that 1 is the bid cap. Thus, players who place bids above 1 have the same chances of obtaining the item as if they were bidding 1 exactly, but they have to pay more because of higher bids. Thus, bidding above 1 is strictly dominated. □

Lemma A.2. *There is no mass of bids in* $(0, r)$

Proof. Suppose by way of contradiction that there is at least one player i, for whom $\mathbb{P}\{\text{bid}_i \in (0, r)\} > 0$. Since r is the reserve price, if the actual bid of player i realizes as $b \in (0, r)$, player i does not obtain the object, but has to pay a positive amount of money. Thus, a strategy of player i that involves bidding in the subset of $(0, r)$ with some non-zero probability is dominated by a similar strategy that replaces the bids in $(0, r)$ with the zero-bid, otherwise remaining the same. □

Lemma A.3. *There are no atoms in* $(r, 1)$*.*

Proof. Suppose, by way of contradiction, that there is at least one player, i, who in equilibrium has an atom of size $P > 0$ at a bid $b_i \in (r, 1)$. Any other player would not have an atom at b_i, since bidding slightly above increases the costs continuously, but causes a discontinuous upward jump in the probability of winning the object. If nobody else besides i bids anywhere in the region of b_i, i would be better off lowering the b_i bid. If there is a player $k \neq i$ who bids with a non-zero probability in the interval $(b_i - \varepsilon, b_i)$, we can choose a small $\varepsilon' \in (0, \varepsilon)$, such that replacing any bid from $(b_i - \varepsilon', b_i)$ to $b_i + \varepsilon$ increases the expenditures by at most $2\varepsilon'$ but increases the probability of winning the object by at least P. We are free to choose ε' so that the net change of k's payoff is positive. Thus, k would have a gap in bids below b_i. This again would cause i to decrease b_i □

Lemma A.4. *There can be no more than one atom at r*

Proof. Suppose, by way of contradiction, that two or more players have atoms at r. Then, any of them would be better off by bidding slightly higher than r. $\square$

Let us introduce the following notation $\bar{b} \doteq \sup\{\cup_i BR_i \cap (r, 1)\}$, $\underline{b} \doteq \inf\{\cup_i BR_i \cap (r, 1)\}$. Then, the following result holds

Lemma A.5. *The set $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\}$ is closed relative to $(\underline{b}, \bar{b})$.*

Proof. If the set $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\}$ is empty, the result holds trivially. If, on the contrary, $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\}$ is non-empty, it means there are at least two players who bid in the interior of $(\underline{b}, \bar{b})$ (it cannot be that only one player bids there, since for him it doing so would increase the expenditures, but not increase the probability of winning). Consider any player $k \in \{1, 2, 3\}$, for whom $BR_k \cap (\underline{b}, \bar{b}) \neq \varnothing$. For all bids b in $BR_k \cap (\underline{b}, \bar{b}) \neq \varnothing$ we can write down $V_k \times \prod_{j \neq k} G_j(b) - b = U_k$, where G_j is a respective CDF of player j's bids, and U_k is k's equilibrium payoff. U_k is a point in $\mathbb{R}$, hence, a closed set. For bids b strictly greater than r, $G_j(b)$ is a continuous function for all $j \in \{1, 2, 3\}$, due to the absence of atoms. Thus, $V_k \times \prod_{j \neq k} G_j(b) - b$ is also a continuous function. For a continuous set, the pre-image of a closed set is a closed set. This establishes the claim of the lemma above. $\square$

Lemma A.6. *The set $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\}$ is an interval.*

Proof. Let $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\}$ be non-empty. Suppose, by way of contradiction, that there is a point b^* in $(\underline{b}, \bar{b})$ that does not belong to a Best Response Set of any of the players. By $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\}$ being closed relative to $(\underline{b}, \bar{b})$, there is an open interval, that contains b^*, any point of which is not in any of the Best Response Sets. Pick the largest such interval and denote it $(\underline{b^*}, \overline{b^*})$. $\overline{b^*}$ must be in the Best Response Set of some player. However, since nobody bids in $(\underline{b^*}, \overline{b^*})$, the probability of winning the item from bidding $\overline{b^*}$ is the same as from bidding $\underline{b^*}$, while $\overline{b^*}$ is more costly. This makes $\overline{b^*}$ less profitable than $\underline{b^*}$, leading to a contradiction. $\square$

Lemma A.7. *If $\cup_i\{BR_i \cap (\underline{b}, \bar{b})\} = (\underline{b}, \bar{b})$ is non-empty, all players have atoms in the set $\{0, r\}$. Moreover, there are at least two players having an atom at 0.*

Proof. Suppose, by way of contradiction, that there is no mass in $\{0, r\}$. That would mean, by absence of atoms in $(r, 1)$, that probability of winning the item from bidding $\underline{b}$ is zero, while the expenditures are strictly positive. That, in turn results in a negative payoff for any player from bidding $\underline{b}$, thus contradicting to $\underline{b}$ being in a Best Response set. The fact that at least two players have atoms at 0 therefore follows from the Lemma A.4. $\square$

Lemma A.8. *For players i and k, $i \neq k$, who have atoms at* 0 $G_i(b) \times V_k = G_k(b) \times V_i$, *for $b \in (\underline{b}, \overline{b})$, being a point of increase of both G_i and G_k*

Proof. Players, who have atoms at 0, have an equilibrium payoff of 0. A point of a increase of a CDF of a player must yield him an equilibrium payoff. So, for the two players, i and k, it holds that

$$V_i G_j(b) G_k(b) - b = 0$$
$$V_k G_j(b) G_i(b) - b = 0,$$

from which the assertion of the lemma follows, given that $G_j(b) > 0$. $\square$

Lemma A.9. *For a player, who has an atom at zero, his CDF is increasing on an interval $(a, \tilde{b}) \subseteq (a, \overline{b})$, $a \geqslant \underline{b}$, this CDF is increasing on $(a, \overline{b})$.*

Proof. Denote by i the player who has an increasing CDF on $(a, \tilde{b})$ and an atom at zero. Suppose, by way of contradiction, that the CDF of player i is constant on $(\tilde{b}, c) \subseteq (\tilde{b}, \hat{b})$, $c > \tilde{b}$. By the Lemma A.6, there is always a pair of players bidding in $(\tilde{b}, c)$, and by the Lemma A.7, at least one of them has an atom at zero, and, hence, a zero equilibrium payoff. Call that player k. From the Lemma A.8, it holds that $G_i(\tilde{b})V_k = G_k(\tilde{b})V_i > 0$. Let $\varepsilon > 0$ be such that at $\tilde{b} + \varepsilon$ player k is still bidding. Then, it must be that for any $x \in (\tilde{b}, \tilde{b} + \varepsilon)$, the payoff of i from bidding such x would be weakly lower than the payoff of k:

$$V_i \prod_{j \neq i} G_j(x) - x \leqslant V_k \prod_{j \neq k} G_j(x) - x \Rightarrow V_i G_k(x) \leqslant V_k G_i(x),$$

which, together with the Lemma A.8 contradicts to G_i being constant on $(\tilde{b}, \tilde{b} + \varepsilon)$ and G_k being strictly increasing there. $\square$

Lemma A.10. *If $(r, \overline{b})$ is non-empty, at least one player, who has an atom at zero, randomizes continuously on $(r, \overline{b})$.*

Proof. A different way to write the statement of the Lemma above is $\underline{b} = r$. Otherwise, bidding anywhere in $(r, \underline{b})$ would be a profitable deviation, since it would yield the same probability of obtaining the item as from bidding $\underline{b}$, but would incur smaller expenditures. $\square$

Lemma A.11. *If $\overline{b} > r$, and if a player has an atom at r, this player randomizes continuously on $(r, \overline{b})$.*

Proof. Denote the player, who has an atom at r by i. The other two players, j and k, must have atoms at zero and, hence, a zero equilibrium payoff. Denote by $u_i^* \geqslant$ the equilibrium payoff of player i. Suppose, by way of contradiction, that player i rejoins bidding at some $\tilde{b} \in (r, \overline{b})$. This implies that bidding in $(r, \tilde{b})$ gives him a weakly lower payoff, than u_i^*. Denote by P the mass that i places at r. From players j and k having a zero equilibrium payoff, it follows that in the interval $(r, \tilde{b})$, the CDFs of j and k are given by $G_j(b) = \frac{b}{P \times V_k}$, and $G_k(b) = \frac{b}{P \times V_j}$. Above $\tilde{b}$, in order for equilibrium payoffs to hold, the CDFs of all three players must given by $G_i(b) = \frac{b\sqrt{V_i}}{\sqrt{V_j V_k (b+u_i^*)}}$, $G_j(b) = \sqrt{\frac{(b+u_i^*)V_j}{V_k V_i}}$, $G_k(b) = \sqrt{\frac{(b+u_i^*)V_k}{V_j V_i}}$. Since there can be no atoms in $(r, 1)$, smooth-pasting must hold: CDFs must be continuous at $\tilde{b}$. That allows to pin down the size of i's atom at r: $P = \frac{\tilde{b}\sqrt{V_1}}{\sqrt{V_j V_k (\tilde{b}+u_i^*}}$. That, in turn, allows to determine i's payoff from an arbitrary bid b in $[r, \tilde{b})$:

$$V_i G_j(b) G_k(b) - b = V_i \frac{b^2}{P^2 V_k V_j} - b = V_i \frac{b^2(\tilde{b} + u_i^*)}{\tilde{b}^2 V_i} - b = \frac{b^2(\tilde{b} + u_i^*)}{\tilde{b}^2} - b.$$

If we plug in $b = r$ into the expression above, we get that the payoff from bidding r is equal to u_i^*, i's equilibrium payoff, only if $\tilde{b} = r$. Otherwise, the payoff from bidding r is lower, and player i would not be willing to put an atom at r, contradicting the condition of the lemma. $\square$

B Appendix

B.1 Completeness of characterization in 1.i and 1.ii.

For the case $V_1 > V_2$, recall the general characterization lemma 1, and also apply the following steps:

- There is no mass strictly above V_2, as player 2 won't ever bid strictly above his valuation.

- 0 is never in a Best Response set of player 1, since he can secure a positive payoff by bidding slightly above V_2.

- player 2 has 0 in the Best Response set.

- A strategy profile according to which player 1 bids r with a unit probability is not an equilibrium. If it was an equilibrium, player 2 would always bid slightly above r outbidding player 1. However, player 1 can always make sure he has a positive payoff.

- $\mathbb{P}\{bid_1 > r\} > 0$. Follows from the point above.

- $\mathbb{P}\{bid_2 \geqslant r\} > 0$. Otherwise, player 1 wouldn't bid above r.

- In equilibrium for both players the subset $BR \cap (r, V_2)$ is closed relative to (r, V_2). As follows from lemma 1.

- $BR_j \cap (r, V_2)$ is an interval for both players, which also follows from lemma 1.

  Denote $\inf(BR_i \cap (r, V_2)) = \underline{b}$, and $\sup(BR_i \cap (r, V_2)) = \overline{b}$.

- $\overline{b} = V_2$.

  Suppose that $\overline{b} < V_2$. Then player 2 can place a bid somewhere between $\overline{b}$ and V_2, win with probability 1 and have a positive payoff. This contradicts to 0 being in his best response set, which was established above.

- For players 1 and 2 distributions of their best responses on $(\underline{b}, V_2)$ have densities and are equal to $\frac{1}{V_2}$ and $\frac{1}{V_1}$, respectively.

For a bid $x \in (\underline{b}, V_2)$ of player j it must hold that $V_j \times \mathbb{P}_{\{bid_k<x\}} - x = u$, where u is player-specific equilibrium payoff. So, for $x, x' : \underline{b} \leqslant x < x' \leqslant V_2$ it holds that $V_j \times \mathbb{P}_{\{bid_k<x\}} - x = V_j \times \mathbb{P}_{\{bid_k<x'\}} - x' \Rightarrow \mathbb{P}_{\{bid_k<x'\}} - \mathbb{P}_{\{bid_k<x\}} = (x'-x)/V_j \Rightarrow \lim_{x' \to x+0} \frac{\mathbb{P}_{\{bid_k<x'\}} - \mathbb{P}_{\{bid_k<x\}}}{x'-x} = 1/V_j$, which establishes existence and expression for the right derivative, with similar logic applying to the left derivative.

- Player 1 has an atom at r.

  Otherwise, player 1 would not be able to reach a unit value of her/his CDF by V_2.

- $\underline{b} = r$, as follows from lemma 1.

- Player 2 has an atom at 0.

  In order for him to reach a unit value of CDF by V_2.

  The sizes of atoms for both players are uniquely pinned down.

For $V_1 = V_2$ the following steps apply:

- There is no mass in $(0, r)$

- There is no mass above $V_1 = V_2 = v$.

- There are no atoms above r.

- There cab be no two atoms at r.

- $\min_i \inf(BR_i) = 0$

- In equilibrium for both players the subset $BR \cap (r, V)$ is closed relative to (r, v).

- $BR \cap (r, v)$ is non-empty. Otherwise, the player with 0 in his best response would be able to have a positive payoff.

- $BR_i \cap (r, v)$ is an interval for both players.

- $BR_1 \cap (r, v) = BR_2 \cap (r, v)$.

- $\sup(BR_1 \cap (r, v)) = \sup(BR_2 \cap (r, V)) = v$. Otherwise, the player with 0 in his best response would be able to have a positive payoff.

- PDFs of distributions of bids of both players in the interval $BR \cap (r, V_2)$ are equal to $\frac{1}{v}$.

- $\inf(BR \cap (r, V)) = r$.

- The fact that CDFs have to reach value of 1 at v, and the value of densities in (r, V) pin down that $\mathbb{P}\left[bid_1 \in \{0, r\}\right] = \mathbb{P}\left[bid_2 \in \{0, r\}\right] = \frac{r}{V}$. Absence of two atoms at r means also that $\mathbb{P}\{bid_1 = r\} \times \mathbb{P}\{bid_2 = r\} = 0$.

B.2 Completeness of characterization in propositions 2.i and 2.ii

Start with the case of $\rho_2^2 V_2 - \rho_2 < \rho_1^2 V_1 - \rho_1$:

- SThere can be no equilibria such that at least one of the players bids 1 with a unit probability.

  If player 1 always bids 1 the only way for player 2 to win the prize is to also bid 1, which will result in player 2 getting the prize with probability ρ_2. This in turn will lead to player 2's payoff of $\rho_2 \times V_2 - 1 \leqslant \times(1 - \rho r) - 1 < 0$. Thus player 2 will bid 0 in response to player 1's bid of 1, hence player 1 has incentives to lower his bid. If player 2 always bids 1, depending on player 1's valuation, he would always bid 0 or 1. In both cases player 2 would be better of lowering his bid (by a small amount, or down to 0, respectively).

- There can be no two atoms at r.

- $\min_{i \in \{1,2\}}(\inf(BR_i)) = 0$.

- A profile such that both players bid only a 0 and a 1 with some probabilities is not an equilibrium. Suppose, by contradiction, there is such an equilibrium profile. It would results in $\mathbb{P}\{\text{bid}_i = 0\} = \frac{1 - \rho_k V_k}{V_k \rho_i}$, which results in profitable deviation to bidding r for any of players.

- For the similar reason, a profile such that one player bids r and 1 only, and the other bids 0 and 1 only, is not an equilibrium. Thus, in equilibrium, the probability of a bid in $(r, 1)$ is greater than zero.

- $BR_i \cap (r, 1)$ is an interval for both players.

- In an equilibrium $BR_1 \cap (r,1) = BR_2 \cap (r,1)$, and $\inf\{BR_i \cap (r,1)\} = r$.

- Densities for players 1 and 2 in interval $(r, \sup(BR \cap 1))$ are equal to $\frac{1}{V_2}$ and $\frac{1}{V_1}$, respectively.

- Either $\sup(BR_1 \cap (r,1)) = \sup(BR_2 \cap (r,1)) = 1$ or $\mathbb{P}\{bid_j = 1\} > 0$ for both players. If it wasn't so, the player with 0 in his best response would be able to achieve a positive payoff by bidding 1 with a higher frequency.

- $\sup(BR_1 \cap (r,1)) = \sup(BR_2 \cap (r,1))$ is equal either to $1/\rho_2 - (\rho_1/\rho_2)V_1$ or to $1/\rho_1 - (\rho_2/\rho_1)V_2$. Let player j have 0 in his best response. Then $\bar{b}$ is determined by him being indifferent between bidding 1 and 0, and $\bar{b}$ and 0:

$$V_j \mathbb{P}\{bid_k < \bar{b}\} - \bar{b} = V_j \left(\rho_j(1 - \mathbb{P}\{bid_k < \bar{b}\}) + \mathbb{P}\{bid_k < \bar{b}\}\right) - 1 = 0.$$

- Only one player has 0 in his best response. Both of them cannot be indifferent between 0 and $1/\rho_2 - (\rho_1/\rho_2)V_1$, or 0 and $1/\rho_1 - (\rho_2/\rho_1)V_2$.

- In equilibrium it is player 2 who has 0 in his best response. Had it been player 1 who has 0 in a best response, $\bar{b}$ would be $1/\rho_2 - (\rho_1/\rho_2)V_1$, from 1's indifference. Then, condition on 1's density and 2's indifference between $1/\rho_2 - (\rho_1/\rho_2)V_1$ and 1 would lead to such values of 1's cdf in $(r, 1/\rho_2 - (\rho_1/\rho_2)V_1)$, that would give a second player a negative payoff from bidding there. He would be better off not including $(r, 1/\rho_2 - (\rho_1/\rho_2)V_1)$ into his BR as he can always secure a payoff of 0.

- $(BR_1 \cap (r,1)) = (BR_2 \cap (r,1)) = (r, 1/\rho_1 - (\rho_2/\rho_1)V_2)$.

- Values of 1's CDF at $(r, 1/\rho_1 - (\rho_2/\rho_1)V_2) \cup 1$ are pinned down by 2's indifference between 0 and $(r, 1/\rho_1 - (\rho_2/\rho_1)V_2) \cup 1$, and by 1's PDF; hence, 1's atom at r is uniquely defined.

- Values of 2's CDF at $(r, 1/\rho_1 - (\rho_2/\rho_1)V_2) \cup 1$ are pinned down by 1's indifference across $[r, 1/\rho_1 - (\rho_2/\rho_1)V_2) \cup 1$, and by 2's PDF; hence, 2's atom at 0 is uniquely defined.

The case $\rho_2^2 V_2 - \rho_2 = \rho_1^2 V_1 - \rho_1$ is similar, but now the upper bound of interior bidding, $\sup(BR \cap (r,1)) = 1/\rho_1 - (\rho_2/\rho_1)V_2$ and $1/\rho_1 - (\rho_2/\rho_1)V_2) = 1/\rho_2 - (\rho_1/\rho_2)V_1$. This makes both players indifferent between bidding at 0 and at 1, resulting in a zero payoff and multiplicity of equilibria due to unidentified value of atom at r; and the identity of the player who would bid r.

C Expenditures in the Two Bidders case

Denote bidder one's expenditures, and bidder two's expenditures as α_1, and α_2, respectively. For the valuations in the region A, $V_2 < 1, V_2 \leqslant V_1$, the expenditures are

$$(\alpha_1; \alpha_2) = \left(\frac{V_2}{2} + \frac{r^2}{2V_2}; \frac{V_2^2 - r^2}{2V_1} \right).$$

In the region A′, $V_1 = V_2 \leqslant 1$, the expenditures are

$$(\alpha_1; \alpha_2) = \left(\frac{V_2}{2} - \frac{r(r - 2t_1)}{2V_2}; \frac{V_2}{2} - \frac{r(r - 2t_2)}{2V_2} \right),$$

where $t_i \in [0, r]$, $t_1 \times t_2 = 0$, representing the multiplicity of equilibria in this region.

In the region B, $1 < V_2 \leqslant \frac{1-\rho_1 r}{\rho_2}$, $\rho_2^2 V_2 - \rho_2 < \rho_1^2 V_1 - \rho_1$, the expenditures are

$$(\alpha_1; \alpha_2) = \left(\frac{\rho_1^2 \left(r^2 + V_2^2\right) + (\rho_2 - \rho_1)(V_2 - 1)^2}{2\rho_1^2 V_2}; \frac{\rho_2^2 V_2^2 + \rho_1 \left(2 - \rho_1 r^2\right) - 1}{2\rho_1^2 V_1} \right).$$

Note that if the tie-breaking rule is fair, $\rho_1 = 0.5$, the expenditures in this region coincide with the expenditures in region A. In B′, the expenditures are

$$\alpha_1 = \frac{V_2(\rho_2^2 V_2 - 2(\rho_2 - \rho_1)) + 1 - \rho_1(2 + r(r - 2t_1)\rho_1)}{2V_2\rho_1^2},$$
$$\alpha_2 = \frac{V_2^2 \rho_2^2 + \rho_1(2 - r(r - 2t_2)\rho_1) - 1}{2V_1\rho_1^2},$$

$t_i \in [0, r]$, $t_1 \times t_2 = 0$.

Finally, in the regions C, C′, and D, the expenditures are:

-

$$(\alpha_1; \alpha) = \left(\frac{V_2 - 1}{\rho_1 V_2}; \frac{V_1 - 1}{\rho_2 V_1} \right),$$

in C, $(V_1, V_2) \in \left[\frac{1-r\rho_2}{\rho_1}, \frac{1}{\rho_1}\right] \times \left[\frac{1-r\rho_1}{\rho_2}, \frac{1}{\rho_2}\right]$;

-

$$(\alpha_1; \alpha_2) = \left(\frac{V_2 - 1 + r \times t(1-\rho_2 V_2)}{\rho_1 V_2}; \frac{1-r}{1-\rho_2 r}\right),$$

$t \in [0,1]$, in C′, $(V_1, V_2) \in \left\{\frac{1-\rho_2 r}{\rho_1}\right\} \times \left[\frac{1-r\rho_1}{\rho_2}, \frac{1}{\rho_2}\right]$;

-

$$(\alpha_1, \alpha_2) = \left(\frac{V_2(1-r\rho_2) + 1 - r}{V_2 \rho_1}, \frac{1-r}{\rho_1 V_1}\right),$$

in D, $(V_1, V_2) \in \left[\frac{1-(1-\rho)r}{\rho}, \infty\right) \times \left[\frac{1-\rho r}{1-\rho}, \frac{1}{1-\rho}\right]$.

D Proofs for the Three Bidders Case

D.1 Proof of completeness of 4.i, for $V_3 < 1$

We know from the Proposition 1 that all the equilibria are characterized by atoms at $\{0\}$, $\{r\}$, and $\{1\}$, and also by continuously distributed mass in $(r, \overline{b})$. Since $V_3 < 1$, player three will never have mass at 1. His equilibrium strategy, therefore, should be characterized by atoms at $\{0\}$, $\{r\}$, and mass in $\left(\underline{b}, \overline{b}\right)$. Moreover, if $\mathbb{P}\{\text{bid}_3 = r\} > 0$ and $\mathbb{P}\{\text{bid}_3 \in (\underline{b}, \overline{b})\} > 0$, then it must be that $\underline{b} = r$. Let us consider the following possibilities: (i) player 3 has mass at $\{0\} \cup (\underline{b}, \overline{b})$, $\underline{b} \geqslant r$; (ii) player 3 has mass at $\{r\} \cup (r, \overline{b})$ and, possibly, mass at $\{0\}$; (iii) player 3 has mass at $\{r\}$ and, possibly, mass at $\{0\}$.

(i) Since player 3 is willing to bid 0, his equilibrium payoff must also be zero. So, for bids $b \in (\underline{b}, \overline{b})$, we can write down:

$$V_3 G_1(b) G_2(b) - b = 0.$$

If $\underline{b} > r$, both of the other two players must be active in $(r, \overline{b})$. So, for $b \in (\underline{b}, \overline{b})$ the following system must hold:

$$V_1G_2(b)G_3(b) - b = u_1^*$$
$$V_2G_1(b)G_3(b) - b = u_2^*$$
$$V_3G_1(b)G_2(b) - b = 0,$$

where $u_i^* \geqslant 0$ are the equilibrium payoffs with $u_1^* \times u_2^* = 0$. If it is the case that $u_2^* = 0$, then it must be that in the region $(\underline{b}, \overline{b})$, $G_2(b) = \sqrt{\frac{(b+u_1^*)V_2}{V_1 \times V_3}}$, $G_3(b) = \sqrt{\frac{(b+u_1^*)V_3}{V_1 \times V_2}}$. G_3 must reach the value of 1 at $\overline{b}$, so we get that $\overline{b} = \frac{V_1 \times V_2}{V_3} - u_1^*$. But evaluating G_2 at $b = \overline{b}$ gives $G_2(\overline{b}) = \frac{V_2}{V_3} > 1$, a contradiction. For the case $u_1^* = 0$ the symmetric argument holds.

If $\underline{b} = r$, and only one of the other two players is active in $(r, \overline{b})$, let player 1 be active in $(r, \overline{b})$. Let the total mass that player 2 has at $\{0\}$ and/or $\{r\}$ be P_2. For bids in $(r, \overline{b})$ it must hold that

$$V_1P_2G_3(b) - b = u_1^*$$
$$V_3P_2G_1(b) - b = 0$$
$$\Rightarrow$$
$$G_3(b) = \frac{b + u_1^*}{P_2V_1}, G_1(b) = \frac{b}{V_3P_2}.$$

G_3 must reach the value of 1 at $\overline{b}$, so $\overline{b} = P_2V_1 - u_1^*$. Check the value of G_1 at $\overline{b}$: $G_1(\overline{b}) = \frac{V_1}{V_3} - \frac{u_1^*}{V_3P_2}$. Since $V_1 > V_3$, it is necessary that player 1 has a strictly positive payoff, $u_1^* > 0$. Moreover, in order for the probability masses of bids of players 1 and 2 to be completely spent and equal to one, players 1 and 2 must have atoms at $b = 1$. Checking that their incentives to bid at 1 are satisfied, we have

$$V_1\left(\frac{1}{2} + \frac{1}{2}P_2\right) - 1 = u_1^*$$
$$V_2\left(\frac{1}{2} + \frac{1}{2}G_1\left(\overline{b}\right)\right) - 1 = 0.$$

Using these conditions to solve for u_1^*, P_2 and $\bar{b}$, we have

$$P_2 = \frac{(V_1-2)V_2}{V_1V_2-2(2-V_2)V_3}, \qquad u_1^* = \frac{(V_1-2)(V_1V_2-(2-V_2)V_3)}{V_1V_2-2(2-V_2)V_3}$$
$$G_1(\bar{b}) = \frac{2-V_2}{V_2}, \qquad \bar{b} = \frac{(V_1-2)(2-V_2)V_3}{V_1V_2-2(2-V_2)V_3}.$$

For the objects to correspond to an equilibrium, it is necessary that $V_2 < 2$. Let us check that player two is weakly worse off from bidding in $(r, \bar{b})$: his payoff from bidding $b = \bar{b}$ is:

$$2 - V_2 - \frac{(V_1-2)(2-V_2)V_3}{V_1V_2-2(2-V_2)V_3} \leqslant 0.$$

If $V_1 > 2$ and $V_1V_2-2(2-V_2)V_3 > 0$, the above condition implies $V_1V_2-2(2-V_2)V_3 < V_1V_3 - 2V_3$, which is not true, given the condition $V_2 > 1 > V_3$. If it is the case that $V_1 < 2$ and $V_1V_2 - 2(2-V_2)V_3 < 0$, it can be seen that these two conditions are incompatible with $\bar{b}$ being weakly lower than V_3 (the latter being necessary for player 3 to be willing to bid up to $\bar{b}$).

(ii) The other two players must have atoms at zero, and, therefore, zero payoffs. $\bar{b}$ cannot be greater than V_3. If there are no atoms at 1, either player one or player two could be better of by bidding slightly below their valuation and winning the item with certainty. If there are atoms at 1, arguments similar to part (i) of this proof apply.

(iii) In order for player 3 to be willing to bid r, both of the other two players must have mass at 0 and zero payoffs. It is not an equilibrium for any of them to bid 0 with probability one, since that would lead to possibility of profitable deviations. Player 3 must have a non-negative payoff from bidding r, so $\mathbb{P}\{\text{bid}_1 = 0\}\mathbb{P}\{\text{bid}_2 = 0\}V_3 = r + u_3^*$, $u_3^* \geqslant 0$. If there is bidding in $(r, \bar{b}) \neq \varnothing$, 1's and 2's indifference with the fact that 3's mass is spent completely on r, by assumption, give

$$G_1(b) = \frac{b}{V_2},$$
$$G_2(b) = \frac{b}{V_1}.$$

Together with absence of atoms in $(r, \bar{b})$ this means that $\mathbb{P}\{\text{bid}_1 = 0\} = \frac{r}{V_2}$, and

$\mathbb{P}\{\text{bid}_2 = 0\} = \frac{r}{V_1}$. Then, 3's payoff from bidding r is $\frac{r^2}{V_1 V_2} V_3 - r$, which is less than zero, given $r < 1$, $V_1 > V_3$, $V_2 > V_3$ and $\min_{i \in \{1,2,3\}}\{V_i\} > r$. This contradicts 3's willingness to bid r.

If there is no bidding in $(r, \bar{b})$, players 1 and 2 must also have atoms at 1. From 1's and 2's indifference between bidding 0 and 1, we have that $\mathbb{P}\{\text{bid}_1 = 0\} = \frac{2-V_2}{V_2}$, $\mathbb{P}\{\text{bid}_2 = 0\} = \frac{2-V_1}{V_1}$. For players 1 and 2 not to be willing to bid slightly above r, it must be that $\frac{2-V_2}{V_2} V_1 \leqslant r$ and $\frac{2-V_1}{V_1} V_2 \leqslant r$. Then, player 3's payoff from bidding r is $\frac{2-V_2}{V_2} \frac{2-V_1}{V_1} V_3 - r \leqslant \frac{r}{V_1} \frac{r}{V_2} V_3 - r < 0$, which contradicts player 3's willingness to bid r.

D.2 Proof of Proposition 4.ii

It is straightforward to check that the CDFs described in the Proposition statement do constitute equilibria. Let us, therefore, show that those equilibria are all equilibria possible for the range of parameters stated in the Proposition. So far we know that all equilibria can be described by atoms at $\{0\}$, $\{r\}$, and $\{1\}$, and continuously distributed mass of bids in $(r, \bar{b})$. Let us follow the steps:

(*i*) There cannot be any mass above $V_2 = V_3 = v$, continuous, or atoms, as for players 2 and 3 it is more expensive to bid above v, than what they can get from the item if they win it. So, $\bar{b} \leqslant v$, in all equilibria.

(*ii*) Player 1 always has a positive equilibrium payoff: he can always bid slightly above v, which makes his payoff bounded below by $V_1 - v > 0$. This implies that player 1 cannot have an atom at 0, since it implies a zero equilibrium payoff.

(*iii*) In equilibrium it is never the case that all the bids concentrate on $\{0\}$ and $\{r\}$. If it was the case, player 2, or player 3 would have a profitable deviation of bidding just slightly above r and winning the item with certainty. Overall, it implies that there is some continuously distributed mass of bids in $(r, \bar{b}) \neq \varnothing$.

(*iv*) $\bar{b} = v$, otherwise either player 2, or player 3 would have a profitable deviation.

(*v*) Since player 1 has a positive payoff, he must be bidding somewhere in $(\underline{b}, v)$, $\underline{b} \geqslant r$. Every point in (r, v) is a point of increase at least for one player. In order to have at least one other player bidding in (r, v), player 1 must have an atom at r. This, in turn, implies that for player 1 every point in (r, v) is a point of increase.

(vi) Player 1's equilibrium payoff is equal to $V_1 - v$. This follows from the fact that the CDFs of bids reach the value of 1 by $b = v$, and at least one of player 2's and player 3's CDF reach value of 1 exactly at $b = v$. Since $b = v$ is in the best response set, the payoff of player 1 from bidding v is his equilibrium payoff and is equal to

$$V_1 \times \underbrace{G_2(v) \times G_3(v)}_{1} - v.$$

To summarize, what we know so far is that player 1 has an atom at r and a continuously distributed mass in (r, v) and a positive payoff of $V_1 - v$. Players 2 and 3 have atoms at 0. At least one of them has a continuously distributed mass in (r, v), while the other could re-join bidding in $(\tilde{b}, v)$, $\tilde{b} \in [r, v]$, or just bid 0 with probability 1. Without loss, we let player 2 bid continuously (r, v). We, therefor, have the following equilibrium conditions:

$$\begin{aligned}
&V_1 G_2(b) G_3(0) - b = V_1 - v \\
&V_2 G_1(b) G_3(0) - b = 0 \\
&V_3 G_1(b) G_2(b) - b \leqslant 0, \text{ for } b \in [r, \tilde{b}), \\
&V_1 G_2(b) G_3(b) - b = V_1 - v \\
&V_2 G_1(b) G_3(b) - b = 0 \\
&V_3 G_1(b) G_2(b) - b = 0, \text{ for } b \in [\tilde{b}, v].
\end{aligned}$$

Taking into account that there is no mass in $(0, r)$, player 1 has an atom at r, players 2 and 3 have atoms at 0, and solving for G_1, G_2, G_3 following from the conditions above, we get the equilibrium CDFs, stated in the proposition, with the $\tilde{b}$ being the "degree of freedom" of this type of equilibria, and causing their multiplicity.

D.3 Proof of Proposition 4.iii and region characterization

Let us first show how the system A was derived.

In the type of equilibrium, such that player 1 has a positive rent, three players bid continuously somewhere in $(\tilde{b}, \hat{b}) \subseteq [r, 1]$, and also have atoms at 1, the following must hold:

$$
\begin{aligned}
& V_1 G_2(b_1) G_3(b_1) - b_1 = U > 0, \forall b_1 \in (\tilde{b}, \hat{b}] \\
& V_1 \left(G_2(\hat{b}) G_3(\hat{b}) + \frac{1}{2}(1 - G_2(\hat{b})) G_3(\hat{b}) + \frac{1}{2} G_2(\hat{b})(1 - G_3(\hat{b})) + \right. \\
& \quad \left. + \frac{1}{3}(1 - G_2(\hat{b}))(1 - G_3(\hat{b})) \right) - 1 = U \\
& V_2 G_1(b_2) G_3(b_2) - b_2 = 0, \forall b_2 \in (\tilde{b}, \hat{b}] \\
& V_2 \left(G_1(\hat{b}) G_3(\hat{b}) + \frac{1}{2}(1 - G_1(\hat{b})) G_3(\hat{b}) + \frac{1}{2} G_1(\hat{b})(1 - G_3(\hat{b})) + \right. \\
& \quad \left. + \frac{1}{3}(1 - G_1(\hat{b}))(1 - G_3(\hat{b})) \right) - 1 = 0 \\
& V_3 G_1(b_3) G_2(b_3) - b_3 = 0, \forall b_3 \in (\tilde{b}, \hat{b}] \\
& V_3 \left(G_1(\hat{b}) G_2(\hat{b}) + \frac{1}{2}(1 - G_1(\hat{b})) G_2(\hat{b}) + \frac{1}{2} G_1(\hat{b})(1 - G_2(\hat{b})) + \right. \\
& \quad \left. + \frac{1}{3}(1 - G_1(\hat{b}))(1 - G_2(\hat{b})) \right) - 1 = 0,
\end{aligned}
$$

which follows from the indifference of players across the increase points for continuous bidding, and also the indifference between bidding 1 and $\hat{b}$, the upper-bound of the continuous bidding of all three players. Since $V_2 = V_3 = v$, it must be that the indifference conditions for players 2 and 3 are identical, and it also means that $G_2(b) = G_3(b)$ for $b \in (\tilde{b}, 1]$. So, after some transformations, we can state that $G_2(b) = G_3(b) = \frac{\sqrt{b+U}}{\sqrt{V_1}}$, $G_1(b) = \frac{b}{v} \times \frac{\sqrt{V_1}}{\sqrt{b+U}}$, for $b \in (\tilde{b}, \hat{b}]$. Since player 2's and 3's indifference conditions are identical, we have two equations instead of three, the condition of players' willingness to bid 1 transforms into

$$
\begin{aligned}
V_1 \left(\frac{1}{3} + \frac{1}{3} \frac{\sqrt{U + \hat{b}}}{\sqrt{V_1}} + \frac{1}{3} \frac{U + \hat{b}}{V_1} \right) - 1 = U \\
v \left(\frac{1}{3} + \frac{1}{6} \frac{\sqrt{U + \hat{b}}}{\sqrt{V_1}} + \frac{\hat{b}}{6v} \times \frac{\sqrt{V_1}}{\hat{b} + U} + \frac{1}{3} \frac{\hat{b}}{v} \right) - 1 = 0,
\end{aligned}
$$

from which it is straightforward to derive system A.

Denoting $t \doteq \sqrt{\frac{U+b}{V_1}}$ and solving for t in the first and the second equation of system

A, we have

$$\begin{cases} t &= \frac{3+2U-\hat{b}}{V_1} - 1 \\ t &= \frac{\frac{\hat{b}}{6v}+\frac{U+\hat{b}}{6V_1}}{\frac{1}{v}-\frac{b}{3v}-\frac{1}{3}}. \end{cases}$$

Since we are interested in finding $U \in \mathbb{R}_+$ and $\hat{b} \in \mathbb{R}_+$, it must be that $t > 0$. So, from the second equation for t it follows that $3 > \hat{b} + v$ must hold. Combining the last inequality with $\hat{b} \geqslant 0$ it must be that the region for this type of equilibria must be included into $v < 3$.

Setting the two expressions for t equal to each other, we have that

$$\frac{3+2U-\hat{b}}{V_1} - 1 = \frac{\frac{\hat{b}}{6v}+\frac{U+\hat{b}}{6V_1}}{\frac{1}{v}-\frac{b}{3v}-\frac{1}{3}}.$$

Expressing U in terms of $\hat{b}$, V_1, and v,

$$U = \frac{12\hat{b} - v\hat{b} - V_1\hat{b} - 2\hat{b}^2 + 6V_1 + 6v - 2V_1 v - 18}{12 - 4\hat{b} - 5v}.$$

Plugging this expression for U back into the first equation of the system (A), above, we have an equation for $\hat{b}$

$$\sqrt{\frac{(6-\hat{b}-2v)V_1 - 6(1-\hat{b})(3-\hat{b}-v)}{(12-4\hat{b}-5v)V_1}} = \frac{2\hat{b}V_1 - v(3-\hat{b}-V_1)}{(12-4\hat{b}-5v)V_1}. \qquad \text{((B))}$$

Raising both sides of the equation to the power of 2, we have a cubic equation in $\hat{b}$,

$$A\hat{b}^3 + B\hat{b}^2 + C\hat{b} + D = 0,$$

where coefficients are

$$\begin{aligned} A &:= -24V_1 \\ B &:= 9v^2 - 42(v-4)V_1 \\ C &:= -3(12V_1(10-V_1) - V_1 v(68-3V_1) + v^2(6+8V_1)) \\ D &:= 9v^2 + 216V_1 - 162vV_1 + 24v^2V_1 - 72V_1^2 + 54vV_1^2 - 9v^2V_1^2. \end{aligned}$$

At least one root of the cubic equations is always a real number. The other two roots are real numbers whenever the cubic discriminant is real. We can check that the root which is always real is greater than 1. Notice that as $\hat{b} \to +\infty$ in the expression $A\hat{b}^3 + B\hat{b}^2 + C\hat{b} + D$ the term $A\hat{b}^3$ becomes dominant. Since $A = -24V_1$ and V_1 is always positive, we get that $A\hat{b}^3 + B\hat{b}^2 + C\hat{b} + D \to -\infty$ as $\hat{b} \to +\infty$. Now, if we plug in the value of $\hat{b} = 1$, we get that $A\hat{b}^3 + B\hat{b}^2 + C\hat{b} + D = A + B + C + D = 9(4 - v)(v - 1)V_1^2$, which is always non-negative given that $v \in (1, 3)$. Hence, the expression $A\hat{b}^3 + B\hat{b}^2 + C\hat{b} + D$ changes the sign, as $\hat{b}$ increases from 1 to infinity. Thus, the root, which is always real, is greater than 1.

So, it is necessary for the kind of equilibrium we are looking for to exist that the cubic discriminant is positive:

$$\mathcal{D} = 18 \times A \times B \times C \times D - 4 \times B^3 \times D + B^2 \times C^2 - 4 \times A \times C^3 - 27 \times A^2 \times D^2 > 0,$$

which can be also simplified to

$$\mathcal{D}^* = -(12v^3 + 49vV_1(8 + 3V_1) - 4v^2(3 + 34V_1) - 4V_1(64 + 3V_1(13 + 6V_1))) > 0. \quad (\mathcal{D}^*)$$

The graph (7) depicts the region of v and V_1 for which the discriminant is positive taking into account also that v must be less than 3. Denote the set of the points $(V_1, v), v \in [1, 3]$ for which the cubic discriminant $\mathcal{D}$ is equal to zero as $\tilde{\Delta}$.

In principle, if there are multiple solutions $(U, \hat{b})$ to the system (A) that all meet the condition $U \geqslant 0$, $0 \leqslant r \leqslant \hat{b} \leqslant 1$, that would mean that there are multiple equilibria candidates of the type we are looking for. From an observation above, there could be at most two real values of $\hat{b}$ that lie below 1. To each of those two real $\hat{b}$ corresponds one real U, as follows from the equation (B). Since we are interested in such pairs $(U, \hat{b})$, that $U \geqslant 0$, finding pairs (V_1, v) such that $U = 0$ is informative for establishing the region, for which the current type of equilibria holds. Setting $U = 0$ in the system (A), we get that it can hold for $V_1 = v$ and also for $V_1 = 3$ when $\hat{b} = 0$. So, at least one U among the solutions to the system (A) is positive for $V_1 \geqslant v$. To see that at least one U from the solution to (A) is positive, note that the expressions there are continuous in all parameters and variables, except for the cases of $v = 0$ or $V_1 = 0$. Then, by setting, for example, $V_1 = 1.75$, $v = 1.5$ numerically solving the system (A) yields approximately

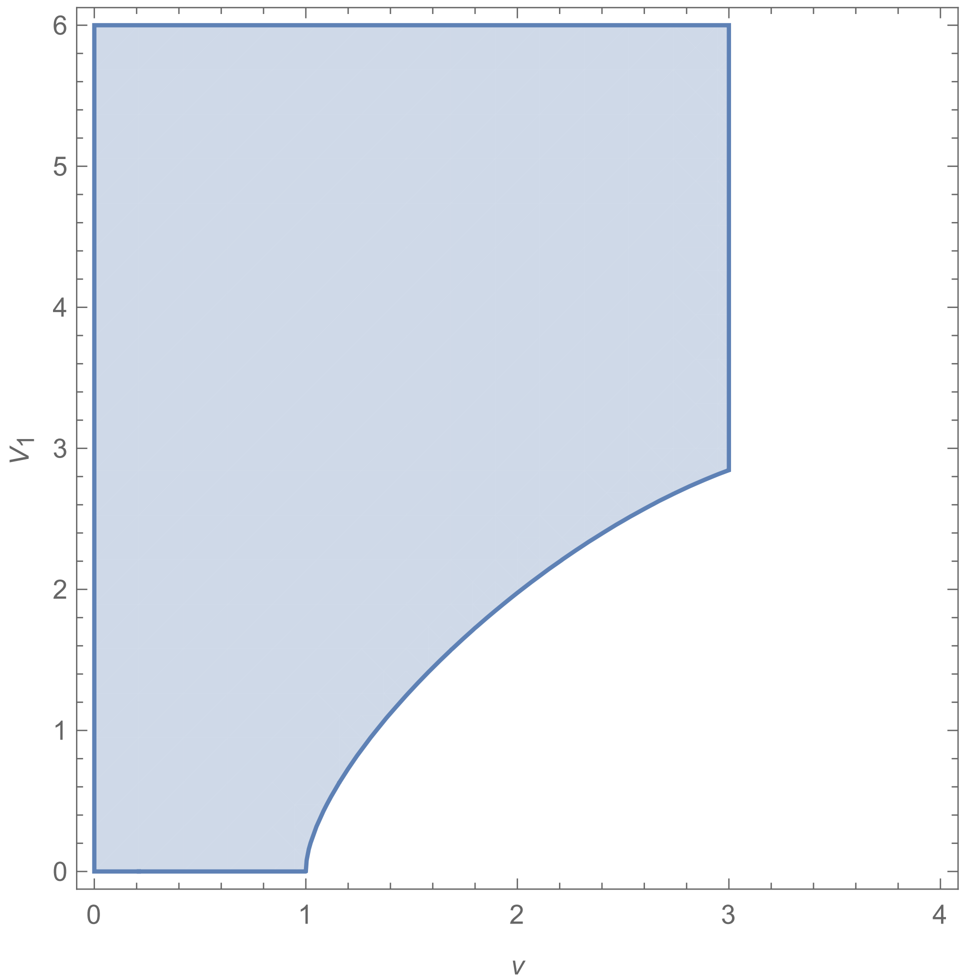


Figure 7: Region for positive value of cubic discriminant

$(U, \hat{b}) = (0.263643, 0.569679)$ and $(U, \hat{b}) = (-0.299109, 0.350856)$.

However, it can also be that the lower of U among the two solutions to the system (A) is equal to 0 at $V_1 = v$. If this is the case, this means that there is a subset of $(V_1, v | V_1 < v)$ where there are two solutions to (A) such that $U > 0$, and, hence, two equilibrium candidates.

To find this subset, notice that the two elements U from the solutions coincide when the two respective $\hat{b}$ coincide. The two solutions $\hat{b}$ coincide when the cubic discriminant defined in the condition $(\mathcal{D}^*)$ is equal to zero. Hence in order to pin down the pair of parameters (V_1, v) for which both solutions U are equal to zero, we must intersect the straight line $V_1 = v$ with the $\tilde{\Delta}$, i.e. set of points that make the value of the cubic discriminant $\mathcal{D}$ equal to zero. Plugging $V_1 = v$ into the expression for $\mathcal{D}^*$ we get $\mathcal{D}^*|_{V_1=v} = -(16 - 7V_1)^2 V_1$. This expression is equal to zero for $V_1 = \frac{16}{7}$ and $V_1 = 0$, but the latter is not in the region we are currently interested in. So, for pairs (V_1, v) such that $\{v < 3, V_1 \geqslant \frac{16}{7}, V_1 \leqslant v\}$, there are two solutions $(U, \hat{b})$ to the system (A) such that the element U is positive.

Overall, in terms of non-negativity of U, one of the necessary conditions for a pair of $(U, \hat{b})$ to be an equilibrium object is that $V_1 \geqslant v$ for $v \in \left[1, \frac{16}{7}\right)$, and $\mathcal{D} \geqslant 0$ for $v \in \left[\frac{16}{7}, 3\right]$.

Another restriction for the solution $(U, \hat{b})$ to correspond to an equilibrium is that $\hat{b} \geqslant r$, i.e. the upper-bound for the continuous part of bids distribution is above the reserve price. To establish the subset of (V_1, v) for which $\hat{b} \geqslant r$, first set $\hat{b} = r$ in the system (A) and solve for the U and V_1. This way we get V_1 expressed in terms of v and r, for which it holds that $\hat{b} = r$. There are two such V_1's:

$$\begin{aligned} V_{1;\mathrm{I,II}} = \frac{1}{3(4-v)(2-r-v)} \Big(& 36 - 60r + 4(7-r)r^2 - 27v + (34 - 7r)rv + 4(1-r)v^2 \\ & \pm (1-r)(12 - 4r - 5v)\sqrt{r^2 - r(6-v) + (3-v)^2} \Big). \end{aligned}$$

The values of $V_{1;\mathrm{I}}$ and $V_{1;\mathrm{II}}$ coincide for three values of v: $v = \frac{4}{5}(3-r)$, $v = \frac{1}{2}\left(6 - r - \sqrt{3r(4-r)}\right)$, and $v = \frac{1}{2}\left(6 - r + \sqrt{3r(4-r)}\right)$. The last value is greater than 3 for $r > 0$, so it is not among the values of (V_1, v) we are interested in. The values of $(4/5)(3-r)$ and $\frac{1}{2}(6 - r - \sqrt{3r(4-r)})$ coincide for $r = \frac{1}{7}$. For $r > \frac{1}{7}$, setting $v = \frac{4}{5}(3-r)$ makes $\sqrt{r^2 - r(6-v) + (3-v)^2}$ not a real number. Thus, for $r \in [0, \frac{1}{7})$, $V_{1,\mathrm{I}}$ and $V_{1,\mathrm{II}}$ intersect twice, while for $r \in [\frac{1}{7}, 1]$ they intersect once, at $v = \frac{1}{2}\left(6 - r - \sqrt{3r(4-r)}\right)$. Notice that for this value of v the value of $V_{1;\mathrm{I}}$ and $V_{1;\mathrm{II}}$ is also $\frac{1}{2}\left(6-r-\sqrt{3r(4-r)}\right)$. This means that the two curves, $V_{1;\mathrm{I}}$ and $V_{1,\mathrm{II}}$ intersect the second time exactly at the 45-degree line.

Consider the curve $V_{1;\mathrm{I}}$. For values of $v<2-r$, the correspondent values of $V_{1;\mathrm{I}}$ are negative, hence, are out of the region of our current interest. There is a discontinuity at $v=2-r$, and $V_{1;\mathrm{I}}$ becomes positive for
$v \in \left(2-r, \frac{1}{2}\left(6-r-\sqrt{3r(4-r)}\right)\right]$. The right limit $\lim_{b\to 2-r+0} V_{1,\mathrm{I}}$ is plus-infinity. This curve crosses the 45-degree line at $v=\frac{8}{3}(1-r)$.

Consider now the curve $V_{1;\mathrm{II}}$. Notice that for $v\to 2-r$ the right limit and the left limit of $V_{1;\mathrm{II}}$ coincide, and, using the L'Hospital's rule, are equal to $\frac{3(2-r)^2}{2(2+r)}$.

Overall, the curves $V_{1;\mathrm{I}}$ and $V_{1;\mathrm{II}}$ in intersect twice for $r<\frac{1}{7}$ and once for $r\geqslant\frac{1}{7}$, at the border of their domain. For $r<\frac{1}{7}$ at $v=2-r$ $V_{1,\mathrm{I}}$ intersects $V_{1,\mathrm{II}}$ from above and then reaches $V_{1,\mathrm{II}}$. Thus, for $r<\frac{1}{7}$ the union of points lying on the curves $V_{1;\mathrm{I}}$ and $V_{1;\mathrm{II}}$ geometrically represent a loop.

Recall that curves $V_{1;\mathrm{I}}$ and $V_{1;\mathrm{II}}$ represent the set of points (V_1, v) for which the element $\hat{b}$ of one of the solutions $(U,\hat{b})$ to the system (A) is equal to r. Recall also that the set $\tilde{\Delta}$ represents the set of points for which the two solutions $(U,\hat{b})$ coincide. So, when the elements $\hat{b}$ of the two solutions coincide *and* are equal to r, the points (V_1,v) from the graph of $V_{1;\mathrm{I}}\cup V_{1;\mathrm{II}}$ and from the graph of $\tilde{\Delta}$ coincide. In other words, the two sets of curves intersect. Notice that they cannot intersect once. This is because given the shape of $V_{1;\mathrm{I}}\cup V_{1;\mathrm{II}}$, it would have to get to the south-east of $\tilde{\Delta}$, in order for the two curves to intersect twice. However, to the south-east of $\tilde{\Delta}$ there are no solutions such that $\hat{b}$ is less than 1, so there it cannot be that $\hat{b}=r$. Thus, the graphs of $V_{1;\mathrm{I}}\cup V_{1;\mathrm{II}}$ and of $\{(V_1,v):\mathcal{D}=0\}$ intersect only once and the graphs are also tangent there. In other words, $(V_{1;\mathrm{I}}\cup V_{1;\mathrm{II}})\cap\tilde{\Delta}$ is a singleton. Denote that point $\left(V_1^{\tilde{\Delta}}, v^{\tilde{\Delta}}\right)$.

Consider the case that the graph of $V_{1;\mathrm{I}}\cup V_{1;\mathrm{II}}$ is a loop, i.e. $r<\frac{1}{7}$. We can distinguish four regions of (V_1,v), as depicted on figure 8 (all regions also imply that the cubic discriminant is non-negative):

(M) $\{(V_1,v)|V_1\leqslant\min\{V_{1;\mathrm{I}};V_{1;\mathrm{II}}\}, v\leqslant v^{\tilde{\Delta}}\}$

(N) $\{(V_1,v)|V_1\in(\min\{V_{1;\mathrm{I}};V_{1;\mathrm{II}}\},\max\{V_{1;\mathrm{I}};V_{1;\mathrm{II}}\}), v<2-r\}$

(N′) $\{(V_1,v)|V_1\in(\min\{V_{1;\mathrm{I}};V_{1;\mathrm{II}}\},\max\{V_{1;\mathrm{I}};V_{1;\mathrm{II}}\}), 2-r\leqslant v\leqslant v^{\tilde{\Delta}}\}$

(K) $(\mathrm{A}\cup\mathrm{B}\cup\mathrm{B}')^C$

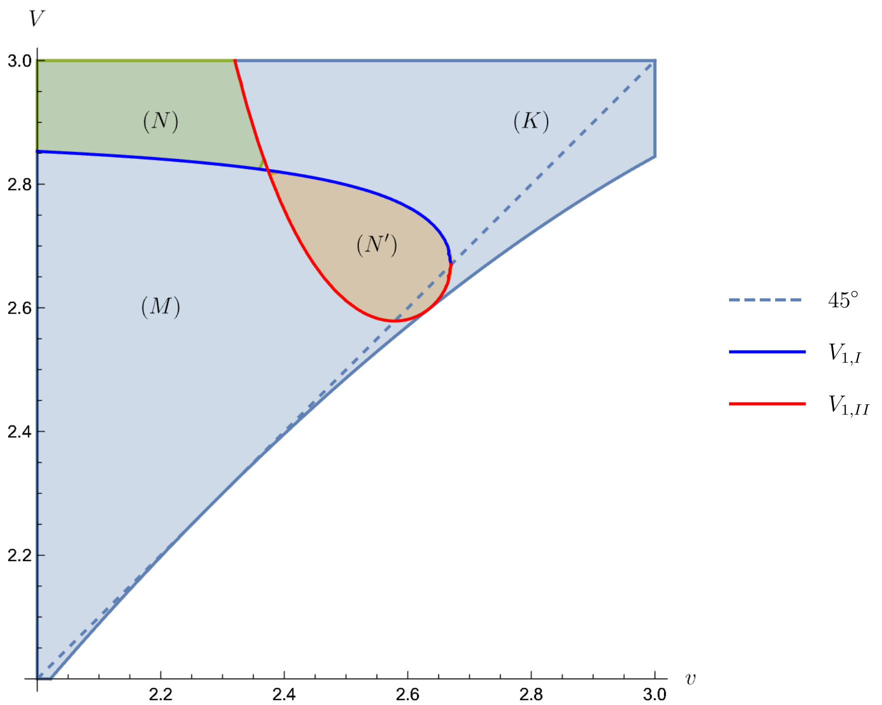


Figure 8: Regions for values of V_1 and v, $r < 1/7$

Recall that the set of points $V_{1;\mathrm{I}} \cup V_{1;\mathrm{II}}$ represent the parameters for which the system (A) has a solution such that $\hat{b} = r$. So, as we cross the graph of $V_{1;\mathrm{I}} \cup V_{1;\mathrm{II}}$ go from the interior of one of the regions above to the interior of the other region, $\hat{b}$ from one of the solutions would cross r, and when we go from the interior of region (M) to the interior of region (K) two solutions would cross r. Let us look at some typical elements of the sets (M)-(K) to understand the ranking of $\hat{b}$ relative to r from the solutions from each of the regions. Set $r = 0.03$. Check that the points $(2.62, 2.2)$, $(2.9, 2.2)$, $(2.62, 2.63)$, $(2.9, 2.63)$ lie in the interiors of regions (M), (N), (N′), and (K), respectively. The correspondent pairs of solutions are:

- $(U, \hat{b}) = (0.526087, 0.125546)$ and $(U, \hat{b}) = (-0.070028, 0.0798147)$ for the point $(V_1, v) = (2.62, 2.2)$;

- $(U, \hat{b}) = (0.809835, 0.0975269)$ and $(U, \hat{b}) = (-0.0201097, 0.020638)$ for the point $(V_1, v) = (2.9, 2.2)$;

- $(U, \hat{b}) = (0.133118, 0.0176779)$ and $(U, \hat{b}) = (0.00617516, 0.0409543)$ for the point $(V_1, v) = (2.62, 2.63)$;

- $(U, \hat{b}) = (0.471844, -0.0553398)$ and $(U, \hat{b}) = (-0.0125482, 0.0138343)$ for the point $(V_1, v) = (2.9, 2.63)$;

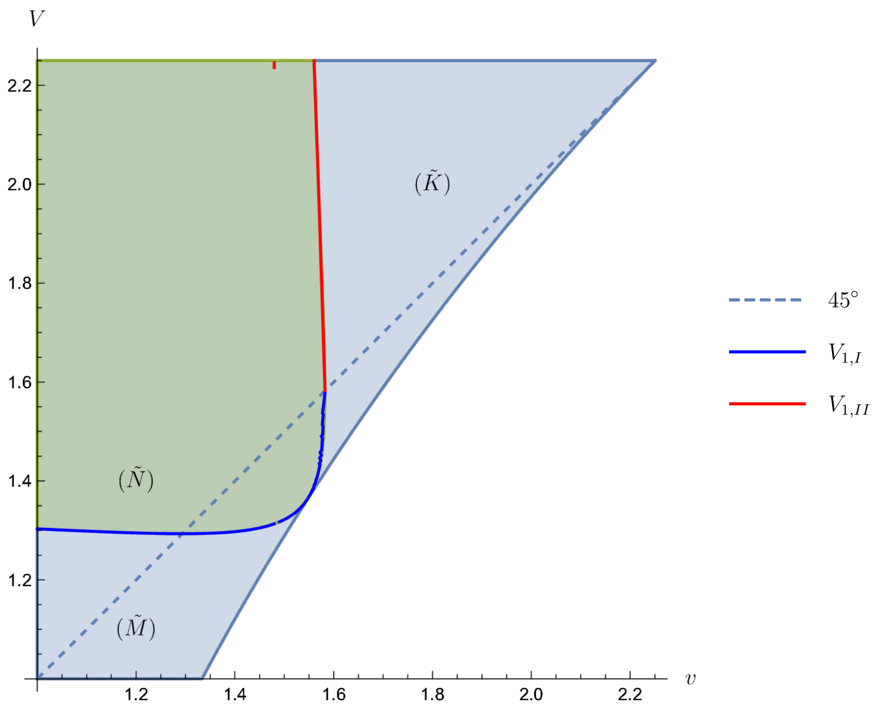


Figure 9: Regions for values of V_1 and v, $r \geqslant 1/7$

Overall, the $\hat{b}$ from the solution that corresponds to a higher U crosses r from above as we go from either of regions (M) and (N) to either of regions (N′) and (K); while the $\hat{b}$ from the solution that corresponds to a low U crosses r from above as we go from either of regions (M) and (N′) to either of regions (N) and (K);

Consider now the case that $r \geqslant \frac{1}{7}$, so that $V_{1,\mathrm{I}}$ and $V_{1,\mathrm{II}}$ intersect only once at $v = \frac{1}{2}\left(6 - r - \sqrt{3r(4-r)}\right)$. For this case we can distinguish only three regions, as depicted on figure 9:

($\tilde{\mathrm{M}}$) $\{(V_1, v) | V_1 \leqslant \min\{V_{1;\mathrm{I}}; V_{1;\mathrm{II}}\}, v \leqslant v^{\tilde{\Delta}}\}$

($\tilde{\mathrm{N}}$) $\{(V_1, v) | V_1 \in (\min\{V_{1;\mathrm{I}}; V_{1;\mathrm{II}}\}, \max\{V_{1;\mathrm{I}}; V_{1;\mathrm{II}}\})$

($\tilde{\mathrm{K}}$) $(\mathrm{A} \cup \mathrm{B})^C$

In this case consider typical elements of those sets. For $r = 0.5$ the points $(V_1, v) = (1.2, 1.1)$, $(V_1, v) = (1.575, 1.1)$, $(V_1, v) = (1.575, 1.65)$ lie in ($\tilde{\mathrm{M}}$), ($\tilde{\mathrm{N}}$), and ($\tilde{\mathrm{K}}$), respectively. The correspondent pairs of solutions are:

- $(U, \hat{b}) = (0.100275, 0.903202)$ and $(U, \hat{b}) = (-0.490029, 0.550519)$ for the point $(V_1, v) = (1.2, 1.1)$;

- $(U,\hat{b}) = (0.475719, 0.902897)$ and $(U,\hat{b}) = (-0.384299, 0.419833)$ for the point $(V_1, v) = (1.575, 1.1)$;

- $(U,\hat{b}) = (-0.0960184, 0.467774)$ and $(U,\hat{b}) = -0.2728940.411679)$ for the point $(V_1, v) = (1.575, 1.65)$;

Thus, the $\hat{b}$ from the solution that corresponds to a higher U crosses r from above as we go from either of regions ($\tilde{\mathrm{M}}$) and ($\tilde{\mathrm{N}}$) to the region ($\tilde{\mathrm{K}}$); and the $\hat{b}$ from the solution that corresponds to a low U crosses r from above as we go from the region ($\tilde{\mathrm{M}}$) to either of regions ($\tilde{\mathrm{N}}$) and ($\tilde{\mathrm{K}}$);

Note also that in the case $r \geqslant \frac{1}{7}$ the curve $V_{1;\mathrm{II}}$ intersects the 45-degree line twice and the larger value of v at the intersection is $\frac{1}{2}\left(6 - r - \sqrt{3r(4-r)}\right)$ (which is also a point at which $V_{1;\mathrm{II}}$ and $V_{1;\mathrm{I}}$ coincide). For $r \geqslant \frac{1}{7}$ the value of $\frac{1}{2}\left(6 - r - \sqrt{3r(4-r)}\right)$ is less than or equal to $\frac{16}{7}$. The point of tangency of $V_{1,\mathrm{I}} \cup V_{1,\mathrm{II}}$ with $\tilde{\Delta}$ (the points for which cubic discriminant is zero), is below the higher intersection of $V_{1,\mathrm{II}}$ with a 45 degree line, which itself is below the point of tangency of a 45 degree line and the set of points $\tilde{\Delta}$. This means that for $r \geqslant \frac{1}{7}$ one of the necessary conditions for a pair $(U,\hat{b})$ to be an equilibrium object is that $V_1 \geqslant v$, since for this region only the solutions with higher U are such that U is non-negative.

Overall, recall that for a pair U, $\hat{b}$ to be an equilibrium object, it needs to be a solution to the system (A), U must be non-negative, and $\hat{b}$ needs to be weakly greater than r. For $r \leqslant \frac{1}{7}$ the equilibrium (equilibria) exists (exist) in the following regions:

(Q) $V_1 \in [v, +\infty)$ for $v \in [1, 2-r]$, $V_1 \in [v, V_{1;\mathrm{I}}]$ for $v \in \left(2-r, \frac{16}{7}\right)$. There is one equilibrium pair U, $\hat{b}$, which corresponds to the solution to the system (A) with a higher U

(Q′) $V_1 \geqslant v$, $v \in \left[\frac{16}{7}, \frac{8}{3}(1-r)\right)$. Similar to the region described above, there is one equilibrium pair $(U,\hat{b})$, which corresponds to the solution to the system (A) with a higher U.

(R) $V_1 < \min\{v, V_{1,\mathrm{I}}\}$, $v \in \left[\frac{16}{7}, \frac{8}{3}(1-r)\right)$, $\mathcal{D} \geqslant 0$. There are two pairs, which solutions to the system (A) and which are also equilibrium objects. This is because the lower U, which is a solution, is now non-negative. And elements $\hat{b}$ which are parts of solution are now not less than r. So, in this sub-region there are two equilibria possible. Note also that player 1 has a positive rent here, $U \geqslant 0$, even though he has lower valuation V_1 than the other two players, who have v.

(S) $V_1 \in [V_{1,\mathrm{I}}, v]$, $v \in \left[\frac{8}{3}(1-r), \frac{1}{2}\left(6-r-\sqrt{3r(4-r)}\right)\right]$. There is one pair, which solutions to the system (A) and which is an equilibrium object. Note, however, that in this pair U is lower, than in the other pair. However, since in the pair with the higher U the element $\hat{b}$ is lower than r, that pair cannot be an equilibrium.

As for $r > \frac{1}{7}$, the equilibrium exists only in the region $V_1 \in [v, +\infty)$ for $v \in [1, 2-r]$, $V_1 \in [v, V_{1;\mathrm{I}}]$ for $v \in \left(2-r, \frac{16}{7}\right)$. An equilibrium pair $(U, \hat{b})$ is the solution to the system (A) with the higher value of U.

D.4 Proposition 4.iv region characterization

Without loss, let player 1 be the one with the non-negative payoff $u_1 \geqslant 0$. Denote the size of player 3's atom at 0 by P_3. The behavior of CDFs of player 1 and player 2 in the continuous part (bidding in $(r, \bar{b})$) follows from

$$V_1 G_2(b) P_3 - b = U$$

$$V_2 G_1(b) P_3 - b = 0$$

$$\Rightarrow$$

$$G_1(b) = \frac{b}{V_2 \times P_3}, G_2(b) = \frac{b+U}{V_1 \times P_3}.$$

The equilibrium objects, that need to be determined, are, therefore, P_3, i.e. with what probability is player 3 inactive; $\hat{b}$, the upper-bound of the support of continuous bidding; and U, player 1's equilibrium payoff. Determining these equilibrium objects and their expressions through V_1, V_2, and V_3 will allow to determine the subset of parameters, for which the current type of equilibrium holds. These objects can be inferred from the indifference of players 1 and 2 between bidding $\hat{b}$ and 1; and indifference of player 3 between bidding 0 and 1. Writing down these conditions:

$$V_1 \times G_2(\hat{b}) \times P_3 - \hat{b} =$$
$$V_1\left(G_2(\hat{b})P_3 + \frac{1}{2}(1 - G_2(\hat{b}))P_3 + \frac{1}{2}G_2(\hat{b})(1 - P_3) + \frac{1}{3}(1 - G_2(\hat{b}))(1 - P_3)\right) - 1 = U,$$
$$V_2 \times G_1(\hat{b}) \times P_3 - \hat{b} =$$
$$V_2\left(G_1(\hat{b})P_3 + \frac{1}{2}(1 - G_1(\hat{b}))P_3 + \frac{1}{2}G_1(\hat{b})(1 - P_3) + \frac{1}{3}(1 - G_1(\hat{b}))(1 - P_3)\right) - 1 = 0,$$
$$V_3\left(G_1(\hat{b})G_2(\hat{b}) + \frac{1}{2}(1 - G_1(\hat{b}))G_2(\hat{b}) + \frac{1}{2}G_1(\hat{b})(1 - G_2(\hat{b})) + \frac{1}{3}(1 - G_1(\hat{b}))(1 - G_2(\hat{b}))\right) - 1 = 0.$$

Using the fact that $G_1(\hat{b}) = \frac{\hat{b}}{V_2 \times P_3}$ and $G_2(\hat{b}) = \frac{b+U}{V_1 \times P_3}$, we can rewrite the above conditions as a system of three equations:

$$\frac{1}{3}V_1 + \frac{1}{6}P_3V_1 + \frac{1}{6}\frac{\hat{b}+U}{P_3} + \frac{1}{3}(\hat{b}+U) = 1 + U \tag{1}$$
$$\frac{1}{3}V_2 + \frac{1}{6}P_3V_2 + \frac{1}{6}\frac{\hat{b}}{P_3} + \frac{1}{3}\hat{b} = 1 \tag{2}$$
$$\frac{1}{3}V_3 + \frac{1}{6}\frac{\hat{b}+U}{V_1P_3} + \frac{1}{6}\frac{\hat{b}}{V_2P_3} + \frac{1}{3}\frac{(\hat{b}+U)\hat{b}}{V_1V_2P_3^2} = 1. \tag{3}$$

Let us denote U and $\hat{b}$ that follow from the equations (1) and (2) as U^* and $\hat{b}^*$, respectively. Then,

$$U^* = \frac{P_3(2+P_3)(V_1 - V_2)}{4P_3 - 1}, \hat{b}^* = \frac{P_3(6 - (2+P_3)V_2)}{1 + 2P_3}.$$

Plugging the expression for $\hat{b}^*$ into the equation (3), solving for U, and denoting that solution as $\tilde{U}$, we get

$$\tilde{U} = \frac{P_3((2+P_3)V_2 - 6)}{1 + 2P_3} + \frac{P_3V_1(6 + V_2(2V_3(1+2P_3) - 8 - 13P_3))}{3(V_2 - 4)}.$$

From now on, let us consider the two sub-cases: (a) $P_3 > \frac{1}{4}$; and (b) $P_3 < \frac{1}{4}$:

(a), $P_3 > \frac{1}{4}$ In this case U^* can only be non-negative if $V_1 \geqslant V_2$. So, this is one of the necessary conditions. Another necessary condition is that $V_2 \leqslant 3$, which follows from the fact that $\hat{b}^*$ must be non-negative. Let us establish the remaining necessary conditions in terms of V_1, V_2, V_3, and r from the facts that: U^* and $\tilde{U}$ must cross and they must be non-negative; $P_3 \in (\frac{1}{4}, 1]$; $\hat{b} \in [r, 1]$; $G_i(\hat{b}) \leqslant 1$, $i = 1, 2$; $V_3 G_1(\hat{b}) G_2(\hat{b}) \leqslant \hat{b}$.

Let us consider U_1^* and $\tilde{U}$ as functions of P_3, $U^*(P_3)$, and $\tilde{U}(P_3)$. We are interested in their intersection (if it exists) at some $P_3^* \in (\frac{1}{4}, 1]$. Since $P_3^* = 0$ is not the intersection we are interested in, we can consider the intersection(s) of the functions $\upsilon^*(P_3) = \frac{\tilde{U}(P_3)}{P_3}$, and $\tilde{\upsilon}(P_3) = \frac{U^*(P_3)}{P_3}$. Note that under the restriction that $V_2 \leqslant 3$, $\tilde{\upsilon}$ is strictly decreasing in V_3. If V_3 is small enough so that $\tilde{\upsilon}(1) \geqslant \upsilon^*(1)$ (i.e. if $V_3 \leqslant \frac{4V_1V_2+5V_2-4-3V_1-V_2^2}{V_1V_2}$), then $\tilde{\upsilon}'(\frac{1}{4}) > 0$ and $\tilde{\upsilon}'(1) > 0$.[11] For $P_3 > 0$, $\tilde{\upsilon}$ can change monotonicity at most once, thus, for $P_3 \in (\frac{1}{4}, 1]$ the two curves cross only once: $\upsilon^*(P_3)$ is decreasing in P_3 with $\lim_{P_3 \downarrow \frac{1}{4}} u^*(P_3) = +\infty$; $\tilde{\upsilon}$ is increasing in P_3 if the two curves cross at $P_3 \in (\frac{1}{4}, 1]$. So, another necessary condition is that $V_3 \leqslant \frac{4V_1V_2+5V_2-4-3V_1-V_2^2}{V_1V_2}$. Note also, that since $\tilde{\upsilon}$ is increasing in V_2 and υ^* is decreasing in V_2 given our restrictions, the point of intersection of the two curves is decreasing in V_2.

Note that $\hat{b}^*$ is always weakly less than 1 under our restrictions. $\hat{b}^*$ needs also to be weakly greater than r when υ^* and $\tilde{\upsilon}$ cross. $\hat{b}^*$ achieves maximum with respect to P_3 at $P_3 = \frac{\sqrt{3}\sqrt{V_2(4-V_2)}}{2V_2} - \frac{1}{2} \in [0, 1]$ for $V_2 \in [1, 3]$. Let us therefore solve for such values of V_3, at which at the intersection of $\tilde{\upsilon}$ and υ^* the value of $\hat{b}^*$ is equal to r, and for values of V_3 between the pairs identified above, it will be satisfied that $\hat{b}^* > r$. Treating, for now, V_3 as endogenous parameter and solving the system

$$\begin{aligned} \tilde{\upsilon}(P_3; V_3) &= \upsilon^*(P_3; V_3) \\ \hat{b}^*(P3) &= r \end{aligned} \tag{C}$$

[11] This is because $\tilde{\upsilon}'(1)$ and $\tilde{\upsilon}'(\frac{1}{4})$ are decreasing in V_3, given our restriction that $V_2 \leqslant 3$; for $\tilde{\upsilon}'(1)$ to be weakly positive, it needs to hold that $V_3 \leqslant \frac{16+V_2(-8+13V_1+V_2)}{4V_1V_2}$, and for $\tilde{\upsilon}'(\frac{1}{4})$ to be weakly positive, it needs to hold that $V_3 \leqslant \frac{13}{4} + \frac{(4-V_2)^2}{V_1V_2}$; both of the latter two inequalities are implied by $V_3 \leqslant \frac{4V_1V_2+5V_2-4-3V_1-V_2^2}{V_1V_2}$.

with respect to P_3 and V_3 gives us two pairs of roots:

$$
\begin{aligned}
P_3^I &= \frac{3-r-V_2-\psi}{V_2} \\
V_3^I &= \frac{1}{V_1V_2(8-8r-3V_2)}\times \\
&\big(16r^3-144+36V_1+V_2(90-13V_2)-V_1V_2(6V_2-2)+ \\
&2r^2(11V_2-56)-r(V_1(12+23V_2)-240+V_2(112-13V_2)) \\
&-(48-12V_1-14V_2+3V_1V_2+16r^2-r(64-14V_2))\psi\big) \\
P_3^{II} &= \frac{3-r-V_2+\psi}{V_2} \\
V_3^{II} &= \frac{1}{V_1V_2(8-8r-3V_2)}\times \\
&\big(16r^3-144+36V_1+V_2(90-13V_2)-V_1V_2(6V_2-2)+ \\
&2r^2(11V_2-56)-r(V_1(12+23V_2)-240+V_2(112-13V_2)) \\
&+(48-12V_1-14V_2+3V_1V_2+16r^2-r(64-14V_2))\psi\big)\,, \qquad (1)
\end{aligned}
$$

where $\psi = \sqrt{(3-r)^2-(6-r)V_2+V_2^2}$. Note that $P_3^I \leqslant P_3^{II}$. Thus, the necessary conditions that follow from the requirement $\hat{b} \geqslant r$ is that $V_3 \geqslant V_3^I$ whenever $P_3^I \in [0.25, 1]$ and $V_3 \leqslant V_3^{II}$ whenever $P_3^{II} \in [\frac{1}{4}, 1]$.

From the restriction $G_1(\hat{b}) \leqslant 1 \iff \frac{\hat{b}}{V_2P_3} \leqslant 1$ follows the necessary condition $V_3 \geqslant \frac{9V_2+14V_1V_2-17V_1-9V_2^2}{V_1(5V_2-8)}$ whenever $\frac{2-V_2}{V_2} \in [\frac{1}{4}, 1]$. This is implied by the following observations: $G_1(\hat{b}^*) = \frac{\hat{b}^*}{V_2P_3} = \frac{6-(2+P_3)V_2}{(1+2P_3)V_2}$ is decreasing in P_3; the P_3 corresponding to the intersection of $\tilde{v}$ and v^* is increasing in V_3; so V_3 must be high enough so that when at the cross of $\tilde{v}$ and v^*, $G_1(\hat{b})$ is weakly less than 1; $G_1(\hat{b})$ is exactly equal to 1 at the intersection of $\tilde{v}$ and v^* when V_3 is equal to $\frac{9V_2+14V_1V_2-17V_1-9V_2^2}{V_1(5V_2-8)}$, and this occurs at $P_3 = \frac{2-V_2}{V_2}$, which establishes the considered condition.

By the similar logic, from the restriction $G_2(\hat{b}) \leqslant 1$ follows the restriction $V_3 \geqslant \frac{32V_2-9V_1-3\left(8-\sqrt{9V_1^2-16(4-V_2)(V_2-1)}\right)}{8V_2}$, whenever $\frac{V_1-4V_2+8-\sqrt{9V_1^2-16(4-V_2)(V_2-1)}}{4(V_1+V_2)} \in [\frac{1}{4}, 1]$ (with the latter being implied by $V_1 \geqslant V_2$).

Finally, it must be non-profitable for player 3 to bid anywhere in $[r, \hat{b}]$, which is formally written down as

$$\begin{aligned}
&V_3 G_1(b) G_2(b) - b \leqslant 0, \forall b \in \left[r, \hat{b}\right] \iff \\
&V_3 \times \frac{b(b+U)}{V_1 V_2 P_3^2} \leqslant b \forall b \in \left[r, \hat{b}\right] \iff \\
&V_3 \times \frac{\hat{b}+U}{V_1 V_2 P_3^2} \leqslant 1.
\end{aligned}$$

Using the expressions for $\tilde{U}$ and $\hat{b}^*$, $V_3 \times \frac{\hat{b}+U}{V_1 V_2 P_3^2}$ can be rewritten as $\frac{V_3(6+V_2(2V_3(1+2P_3)-8-13P_3))}{3(V_2-4)V_2 P_3} \doteq \phi(P_3)$. Note that if $\phi(P_3) = 1$ for some $P_3^* \in [\frac{1}{4}, 1]$, then $\phi(P_3)' \geqslant 0$.[12] Using the fact that the point of intersection of υ^* and $\tilde{\upsilon}$, $P_3^\dagger$, is increasing in V_1,[13], we can establish the upper bound on V_1, such that if V_1 is below that bound, $V_3 G_1(\hat{b}) G_2(\hat{b}) \leqslant \hat{b}$. That bound follows from the system

$$\begin{aligned}
\upsilon^*(P_3) &= \tilde{\upsilon}(P_3) \\
V_3 G_1(\hat{b}^*) G_2(\hat{b}^*) &= \hat{b}^*,
\end{aligned}$$

and is

$$\begin{aligned}
\overline{V_1} = &\frac{1}{(V_2-V_3)^2(36+V_2^2(V_3-1)+V_3(4V_3-13)-V_2(5+V_3(43+V_3(4V_3-25))))} \times \\
&\times \left((V_2(3V_2-12+(13-4V_3)V_3)(V_3((23-8V_3)V_3-24)+\right. \\
&\left. V_2^2(3+4(V_3-4)V_3)-2V_2(6+V_3(2V_3(16+(V_3-7)V_3)-35)))\right).
\end{aligned}$$

Combining all the necessary conditions established above, we have the necessary and sufficient condition for the type of equilibrium with players 1 and 2 bidding continuously in $(r, \hat{b})$, and having atoms at 0, r, and 1; while player 3 only has atoms at 0 and 1, with

[12] To see this, note that the root of $\phi(P_3) = 1$ is $P_3^* = \frac{2(3-V_2(4-V_3))V_3}{V_2((13-4V_3)V_3+3V_2-12)}$; while the derivative of $\phi(P_3)$ is $\phi(P_3)' = \frac{2(3-V_2(4-V_3))V_3}{3P_3^2(4-V_2)V_2}$. The minimum of that derivative, given the constraints $P_3^* \in [\frac{1}{4}, 1]$ is 0; thus $\phi(P_3)$ is weakly increasing in the parameters region of our interest.

[13] As can be seen by taking the implicit derivative P_3 with respect to V_1 that follows from $F(P_3) = \upsilon^* - \tilde{\upsilon}$.

the size of atom at 0 being greater than $\frac{1}{4}$:

$$
\begin{aligned}
& V_1 \geqslant V_2 \\
& V_2 \leqslant 3 \\
& V_3 \leqslant \frac{4V_1V_2 + 5V_2 - 4 - 3V_1 - V_2^2}{V_1V_2} \\
& V_3 \geqslant \frac{1}{V_1V_2(8 - 8r - 3V_2)} \times \\
& \quad \big(16r^3 - 144 + 36V_1 + V_2(90 - 13V_2) - V_1V_2(6V_2 - 2) + \\
& \quad 2r^2(11V_2 - 56) - r(V_1(12 + 23V_2) - 240 + V_2(112 - 13V_2)) \\
& \quad -(48 - 12V_1 - 14V_2 + 3V_1V_2 + 16r^2 - r(64 - 14V_2))\psi\big), \\
& \quad \text{when } \frac{3 - r - V_2 - \psi}{V_2} \in [\frac{1}{4}, 1] \\
& V_3 \leqslant \frac{1}{V_1V_2(8 - 8r - 3V_2)} \times \\
& \quad \big(16r^3 - 144 + 36V_1 + V_2(90 - 13V_2) - V_1V_2(6V_2 - 2) + \\
& \quad 2r^2(11V_2 - 56) - r(V_1(12 + 23V_2) - 240 + V_2(112 - 13V_2)) \\
& \quad +(48 - 12V_1 - 14V_2 + 3V_1V_2 + 16r^2 - r(64 - 14V_2))\psi\big), \\
& \quad \text{when } \frac{3 - r - V_2 + \psi}{V_2} \in [\frac{1}{4}, 1] \\
& V_3 \geqslant \frac{9V_2 + 14V_1V_2 - 17V_1 - 9V_2^2}{V_1(5V_2 - 8)} \\
& V_3 \geqslant \frac{32V_2 - 9V_1 - 3\left(8 - \sqrt{9V_1^2 - 16(4 - V_2)(V_2 - 1)}\right)}{8V_2} \\
& V_1 \leqslant \frac{1}{(V_2 - V_3)^2(36 + V_2^2(V_3 - 1) + V_3(4V_3 - 13) - V_2(5 + V_3(43 + V_3(4V_3 - 25))))} \times \\
& \quad \times \big((V_2(3V_2 - 12 + (13 - 4V_3)V_3)(V_3((23 - 8V_3)V_3 - 24) + \\
& \quad V_2^2(3 + 4(V_3 - 4)V_3) - 2V_2(6 + V_3(2V_3(16 + (V_3 - 7)V_3) - 35))))\big),
\end{aligned}
$$

where $\psi = \sqrt{(3 - r)^2 - (6 - r)V_2 + V_2^2}$.

(b), $P_3 < \frac{1}{4}$. In this sub-case, the exact algebraic expression for the region is not presented. However, the procedure, that can used to derive the respective region is described. Recall the expressions for U^*, and $\tilde{U}$:

$$\begin{aligned} U^* &= \frac{P_3(2+P_3)(V_1-V_2)}{4P_3-1}, \\ \tilde{U} &= \frac{P_3((2+P_3)V_2-6)}{1+2P_3} + \frac{P_3V_1(6+V_2(2V_3(1+2P_3)-8-13P_3))}{3(V_2-4)}. \end{aligned}$$

The root of $U^* = \tilde{U}$, $P_3 = 0$ is not a root we are interested in, as well as the cases $P_3 = \frac{1}{4}$, or $P_3 = -\frac{1}{2}$. In lieu of $U^* = \tilde{U}$, we can therefore work with the cubic equation in P_3:

$$\begin{aligned} 3(2+P_3)(V_1-V_2)(1+2P_3)(V_2-4) = 3((2+P_3)V_2-6)(4P_3-1)(V_2-4)+ \\ +V_1(6+V_2(2V_3(1+2P_3)-8-13P_3))(4P_3-1)(1+2P_3). \end{aligned}$$

The root that is real and always exists is negative. Therefore for there to exist a root $P_3 \in (0, 1/4)$, it is necessary that a cubic discriminant is positive. Using the notation,

$$\begin{aligned} A &= 4V_1V_2(13-4V_3) \\ B &= -3(3(-4+V_2)V_2+4V_1(3+V_2(-4+V_3))) \\ C &= 9(4+V_1-2V_2)(-4+V_2) \\ D &= -9(-4+V_1+V+2)+V_1V_2(-1+V_3), \end{aligned}$$

we have that a necessary condition for the equilibrium with $P_3 \in (0, 1/4)$ to exist is

$$18 \times A \times B \times C \times D - 4 \times B^3 \times D + B^2 \times C^2 - 4 \times A \times C^3 - 27 \times A^2D^2 \geqslant 0.$$

From the fact that we need $\hat{b} \geqslant r \geqslant 0$, and using the expression for $\hat{b}^\star$,

$$\hat{b}^\star = \frac{P_3(6-(2+P_3)V_2)}{1+2P_3},$$

it follows that it is necessary that $V_2 \leqslant 3$.

Under the restrictions that $1 \leqslant V_1 \leqslant V_2 \leqslant V_3$, the function $\tilde{v}(P_3) = \frac{\tilde{U}(P_3)}{P_3}$ is increasing in P_3 on $P_3 \in (0, 1/4)$. If it was decreasing in P_3, or if it achieved a maximum for some $P_3 = P^* \in (0, 1/4)$, the function $\tilde{U}$ would be negative on $P_3 \in (0, 1/4)$, contradicting the equilibrium condition of non-negative players' payoffs. We, therefore, must require that

$\tilde{U}(1/4) \geqslant 0$, which can be written down as

$$-16 + 6V_2 - \frac{(V_1(8 + V_2(-15 + 4V_3)))}{4 - V_2} \geqslant 0.$$

Besides the restriction that $\hat{b} \geqslant 0$, we need to impose more requirement in order to insure $\hat{b} \geqslant r$. Similarly to as we did before, in case $P_3 > 1/4$, we need to consider roots (P_3, V_3) of the system C. However, now we need to account for the fact that there are up to two intersections of U^* and $\tilde{U}$ possible. The P_3 that follows from higher intersection of U^* and $\tilde{U}$ is decreasing in V_3 and the P_3 that follows from the lower intersection of U^* and $\tilde{U}$ is increasing in V_3. There are also two intersections of $\hat{b}^* = r$ with respect to P_3, and those values of P_3 that lie between the points of intersection are the ones for which the condition $\hat{b}^* \geqslant r$ is satisfied. From the expressions of the roots of the system, (P_3^I, V_3^I), and (P_3^{II}, V_3^{II}) it is possible to say that P_3^{II} corresponds to the higher intersection of $\tilde{b}^* = r$ and P_3^I - to the lower. But it is not possible to say whether either of V_3^I or V_3^{II} corresponds to the lower or higher intersection of $\tilde{U}$ and U^*. However, an implicit derivative of P_3 that follows from $\tilde{U} = U^*$ with respect to V_3 can be evaluated at $(P_3, V_3) = (P_3^I, V_3^I)$ and $(P_3, V_3) = (P_3^{II}, V_3^{II})$. The parameters with $\frac{dP_3}{dV_3} > 0$ would correspond to the lower intersection of $\tilde{U}$ and U^*, and vice versa. Using the implicit function derivative,

$$\begin{aligned}
\frac{dP_3}{dV_3} &= -\frac{\frac{\partial(\tilde{U}-U^*)}{\partial V_3}}{\frac{\partial(\tilde{U}-U^*)}{\partial P_3}} \\
&= \frac{2(2P_3+1)V_1V_2}{3(V_2-4)\left(\frac{12(P_3(P_3(7V_2-16)+V_2+8)+V_2-1)}{\left(8P_3^2+2P_3-1\right)^2} + \frac{2V_1(V_2(-4P_3(2P_3-1)(4V_3-13)-2V_3-7)+54)}{3(1-4P_3)^2(V_2-4)}\right)} \\
&\doteq \chi(P_3; V_3).
\end{aligned}$$

Then, the restrictions that follow from $\hat{b} \geqslant r$ are:

- for the region of parameters (V_1, V_2, V_3, r) such that $P_3^I > 1/4$ and $P_3^{II} > 1/4$, there are no subsets such that $\hat{b} \geqslant r$;

- for the region of parameters (V_1, V_2, V_3, r) such that $P_3^I < 1/4$ and $P_3^{II} > 1/4$, the subset of parameters such that $\chi(P_3^I, V_3^I) < 0$ (i.e. the parameters for which (P_3^I, V_3^I) corresponds to the higher intersection of U^* and $\tilde{U}$), we need to exclude

$V_3 > V_3^I$; remainder of the region $P_3^I < 1/4$ and $P_3^{II} > 1/4$ can be included;

- for the region of parameters (V_1, V_2, V_3, r) such that $P_3^I < 1/4$ and $P_3^{II} < 1/4$,
  - the subset of parameters for which $\chi(P_3^I; V_3^I) < 0$ and $\chi(P_3^{II}; V_3^{II}) < 0$ should be included entirely;
  - the subset $\chi(P_3^I; V_3^I) < 0$ and $\chi(P_3^{II}; V_3^{II}) > 0$ is empty, since $P_3^I < P_3^{II}$, but $\chi < 0$ means higher intersection of $\tilde{U}$ and U^*;
  - from the subset for which $\chi(P_3^I; V_3^I) > 0$ and $\chi(P_3^{II}; V_3^{II}) < 0$ exclude $V_3 < \min\{V_3^I, V_3^{II}\}$;
  - from the subset for which $\chi(P_3^I; V_3^I) > 0$ and $\chi(P_3^{II}; V_3^{II}) > 0$ exclude $V_3 \in ([V_3^I, V_3^{II}])^C$.

The remaining restrictions follow from making $G_1(\hat{b}) \leqslant 1$, $G_2(\hat{b}) \leqslant 1$, and making sure it is non-profitable for player 3 to bid anywhere in $b \in (0, 1)$.

In equilibrium of this type, $G_1(\hat{b}) = \hat{b}/(V_2 P_3)$. Replacing $\hat{b}$ with the expression for $\hat{b}^*$, and treating it like a function of P_3, we have that

$$G_1(P_3) = \frac{6 - (2 + P_3)V_2}{V_2(1 + 2P_3)}.$$

This expression is decreasing in P_3 under the restriction $V_2 \leqslant 3$ that we have previously established. $G_1(P_3) = 1$ at $P_3 = \frac{2}{V_2} - 1$. Since we need to have $G_1 \leqslant 1$ for some $P_3 \leqslant 1/4$, we need to impose $V_2 \geqslant \frac{8}{5}$. In order to insure that $G_1 \leqslant 1$ for the intersection of $\tilde{U}$ and U^*, let us solve the system

$$\begin{aligned} G_1(P_3) &= 1 \\ \tilde{U} &= U^* \end{aligned}$$

with respect to P_3 and V_3. We get that the root of that system is $P_3^\dagger = \frac{2}{V_2} - 1$ and $V_3^\dagger = \frac{V_1(17 - 14V_2) + 9(V_2 - 1)V_2}{V_1(8 - 5V_2)}$. If $P_3^\dagger$ is below 0 (happens for $V_2 \geqslant 2$), no additional restrictions are needed. If $P_3^\dagger \in (0, 1/4)$, we need to insure that equilibrium $P_3 \geqslant P_3^\dagger$, since $G_1(P_3)$ is decreasing in P_3 for at least one of the two non-trivial roots of $\tilde{U} = U^*$. $G_1(P_3)$ does not depend on V_3. As we saw, equilibrium P_3 can increase or decrease in V_3, depending on whether it corresponds to the higher or the lower intersection of $\tilde{U}$ and U^*. Check how many roots of $U^* = \tilde{U}$ there are in $P_3 \in (0, 1/4)$. If $2(V_2 - V_1) \leqslant \frac{2V_1(V_2(V_3-4)+3)}{3(V_2-4)} + 2V_2 - 6$, there is one root, that is the one which is decreasing in V_3. For this case we need to restrict $V_3 \leqslant V_3^\dagger$. If $2(V_2 - V_1) > \frac{2V_1(V_2(V_3-4)+3)}{3(V_2-4)} + 2V_2 - 6$, there are two roots. For this case we need to understand the sign of $\frac{dP_3}{dV_3}$. Again using the function $\chi(P_3, V_3)$, we need to have $V_3 \leqslant V_3^\dagger$ if $\chi(P_3^\dagger, V_3^\dagger) < 0$.

Consider the restriction $G_2(\hat{b}) \leqslant 1$. In equilibrium $G_2(\hat{b}) = \frac{U+\hat{b}}{V_1 P_3}$. Using the expressions for $\tilde{U}$ and $\hat{b}^*$, G_2 can be rewritten, as a function of P_3:

$$G_2(P_3) = \frac{V_2(4P_3V_3 - 13P_3 + 2V_3 - 8) + 6}{3(V_2 - 4)},$$

which is linear in P_3, and, besides, does not depend on V_1. Given that we are already restricted to $V_2 \leqslant 3$, G_2 is increasing in P_3 if $V_3 \leqslant 13/4$ and vice versa. Solving the system

$$\begin{aligned} U^* &= \tilde{U} \\ G_2(P_3) &= 1 \end{aligned}$$

with respect to P_3 and V_1, we get the root

$$\begin{aligned} P_3^\ddagger &= \frac{2V_2V_3 - 11V_2 + 18}{13V_2 - 4V_2V_3} \\ V_1^\ddagger &= \frac{V_2(V_2(-4(V_3 - 8)V_3 - 55) - 24V_3 + 51)}{9(V_2(V_3 - 4) + 3)}. \end{aligned}$$

So, if ($V_3 \leqslant 13/4$ and $P_3^\ddagger \geqslant 1/4$), or if ($V_3 \geqslant 13/4$ and $P_3^\ddagger \leqslant 0$), there are no additional restrictions. We need to exclude all the parameters such that ($V_3 \leqslant 13/4$ and $P_3^\ddagger \leqslant 0$), or ($V_3 \geqslant 13/4$ and $P_3^\ddagger \geqslant 1/4$). For $P_3^\ddagger \in [0, 1/4]$, we need to also consider the number of roots of $\tilde{U} = U^*$ and, potentially, the sign of $\frac{dP_3}{dV_1}$ of the P_3 that follows from $\tilde{U} = U^*$, derivative being evaluated at $(P_3^\ddagger, V_1^\ddagger)$. If the there is one non-trivial root of $\tilde{U} = U^*$ in $P_3 \in (0, 1/4)$, it is the root that is increasing in V_1. Then V_1 must be $\leqslant V_1^\ddagger$, if $V_3 \leqslant 13/4$, and V_1 must be $\geqslant V_1^\ddagger$ if $V_3 \geqslant 13/4$. For the case of two non-trivial roots of $U^* = \tilde{U}$ in $(0, 1/4)$, determine the sign of $\frac{dP_3}{dV_1}$ at $(P_3, V_1) = (P_3^\ddagger, V_1^\ddagger)$. If $\frac{dP_3}{dV_1}|_{(P_3^\ddagger, V_1^\ddagger)} \leqslant 0$ AND $V_3 \leqslant 13/4$, V_1 must be weakly below $V_1^\ddagger$, while if $\frac{dP_3}{dV_1}|_{(P_3^\ddagger, V_1^\ddagger)} \geqslant 0$ AND $V_3 \geqslant 13/4$, V_1 must be weakly above $V_1^\ddagger$. The sign of $\frac{dP_3}{dV_1}$ is determined from the equation $\tilde{U} = U^*$ using the implicit function derivative:

$$
\begin{aligned}
\frac{dP_3}{dV_1} &= -\frac{\frac{\partial(U^*-\tilde{U})}{\partial V_1}}{\frac{\partial(U^*-\tilde{U})}{\partial P_3}} = \\
&= -\frac{2(2P_3+1)(P_3V_2(4V_3-13)+V_2(-V_3)+V_2+9)}{3(4P_3-1)(V_2-4)\left(\frac{12(P_3(P_3(7V_2-16)+V_2+8)+V_2-1)}{\left(8P_3^2+2P_3-1\right)^2}+\frac{2V_1(V_2(-4P_3(2P_3-1)(4V_3-13)-2V_3-7)+54)}{3(1-4P_3)^2(V_2-4)}\right)} \\
&\doteq \delta(P_3;V_1).
\end{aligned}
$$

The last remaining restriction is from non-profitability of player 3's deviation to bidding in the interior of $(0,1)$, which happens if and only if $V_3G_1(\hat{b})G_2(\hat{b}) \leqslant \hat{b}$. The latter can be rewritten as $V_3\frac{\hat{b}+U}{V_1V_2P_3^2} \leqslant 1$, or

$$
\tau(P_3) \doteq \frac{V_3(V_2(4P_3V_3-13P_3+2V_3-8)+6)}{3P_3(V_2-4)V_2} \leqslant 1.
$$

$\tau(P_3)$ is increasing whenever $3-V_2(4-V_3) \geqslant 0$ and vice versa.

Let us solve for the roots (P_3, V_1) of the system

$$
\begin{aligned}
\frac{V_3(V_2(4P_3V_3-13P_3+2V_3-8)+6)}{3P_3(V_2-4)V_2} &= 1 \\
U^* &= \tilde{U},
\end{aligned}
$$

and get the root

$$
\begin{aligned}
P_3^{\S} &= \frac{2V_3(V_2(V_3-4)+3)}{V_2(3V_2+V_3(13-4V_3)-12)} \\
V_1^{\S} &= (V_2(3V_2+V_3(13-4V_3)-12)\left(V_2^2(4(V_3-4)V_3+3)\right. \\
&\quad - 2V_2(V_3(2V_3((V_3-7)V_3+16)-35)+6) \\
&\quad + V_3((23-8V_3)V_3-24)))/\left((V_2-V_3)^2\left(V_2^2(V_3-1)-\right.\right. \\
&\quad - \left.\left. V_2(V_3(V_3(4V_3-25)+43)+5)+V_3(4V_3-13)+36\right)\right).
\end{aligned}
$$

As before, we need to check for the number of roots of $U^* = \tilde{U}$, and the sign of $\frac{dP_3}{dV_1}|_{(P_3^{\S},V_1^{\S})}$. If ($P_3^{\S} \geqslant 1/4$ and $3+V_2(V_3-4) \geqslant 0$), there are no new restrictions needed, all parameters such that ($P_3^{\S} \geqslant 1/4$ and $3+V_2(V_3-4) \leqslant 0$) should be excluded. If there is one non-trivial root of $\tilde{U} = U^*$ in $(0,1/4)$, this is the one that is increasing in V_1. So if ($P_3^{\S} \leqslant 1/4$ and

$3+V_2(V_3-4) \geqslant 0$), we must exclude $V_1 > V_1^{\S}$; while if ($P_3^{\S} \leqslant 1/4$ and $3+V_2(V_3-4) \leqslant 0$), we must exclude $V_1 < V_1^{\S}$. If there are two roots of $\tilde{U} = U^*$ in $(0, 1/4)$, determine the sign of $\frac{dP_3}{dV_1}$ at $(P_3, V_1) = (P_3^{\S}, V_1^{\S})$. If $\frac{dP_3}{dV_1}|_{(P_3^{\S}, V_1^{\S})} \geqslant 0$ AND $3 + V_2(V_3 - 4) \leqslant 0$, V_1 must be weakly above $V_1^{\S}$, while if $\frac{dP_3}{dV_1}|_{(P_3^{\S}, V_1^{\S})} \leqslant 0$ AND $3 + V_2(V_3 - 4) \geqslant 0$, V_1 must be weakly below $V_1^{\S}$. The sign of the derivative $\frac{dP_3}{dV_1}$ again can be determined using the function $\delta(P_3; V_1)$.

References

Baye, Michael R, Dan Kovenock, and Casper G De Vries, "Rigging the lobbying process: an application of the all-pay auction," *The american economic review*, 1993, *83* (1), 289–294.

_, _, and _, "The all-pay auction with complete information," *Economic Theory*, 1996, *8* (2), 291–305.

Boleslavsky, Raphael and Christopher Cotton, "Limited capacity in project selection: Competition through evidence production," *Economic Theory*, 2018, *65* (2), 385–421.

Che, Yeon-Koo and Ian L Gale, "Caps on political lobbying," *The american economic review*, 1998, *88* (3), 643–651.

_ and _, "Optimal design of research contests," *American Economic Review*, 2003, *93* (3), 646–671.

Hillman, Arye L and John G Riley, "Politically contestable rents and transfers," *Economics & Politics*, 1989, *1* (1), 17–39.

Letina, Igor, Shuo Liu, and Nick Netzer, "Delegating performance evaluation," *Theoretical Economics*, 2020, *15* (2), 477–509.

Manez, Juan A, María E Rochina-Barrachina, Amparo Sanchis, and Juan A Sanchis, "The role of sunk costs in the decision to invest in R&D," *The Journal of Industrial Economics*, 2009, *57* (4), 712–735.

Moldovanu, Benny and Aner Sela, "Patent licensing to Bertrand competitors," *International Journal of Industrial Organization*, 2003, *21* (1), 1–13.

Muratov, Oleg, "Essays on Information Design in Economic Theory," 2019.

—, "Mapping an Information Design Game into an All-Pay Auction," *Mimeo*, 2021.

Siegel, Ron, "All-Pay Contests," *Econometrica*, 2009, *77*, 71–92.

—, "Asymmetric all-pay auctions with interdependent valuations," *Journal of Economic Theory*, 2014, *153*, 684–702.

Szech, Nora, "Tie-breaks and bid-caps in all-pay auctions," *Games and Economic Behavior*, 2015, *92*, 138–149.