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Fighting for Lemons: The Encouragement Effect in Dynamic Contests with Private Information*

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Abstract

This paper proposes a tractable model of a dynamic contest where players have private information about the contest's prize. We show that private information helps to encourage players who have fallen behind, leading to an increase in aggregate incentives. We derive the optimal information design for a designer interested in the maximization of aggregate effort. Optimal signals turn out to be private and imperfectly informative and aim to level the playing field at any stage of the dynamic interaction.

Keywords: Dynamic contests, discouragement effect, information design.

JEL: C72, D72, D82.

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"But many that are first shall be last; and the last shall be first." (Matthew 19:30 King James Bible.)

1 Introduction

Contests are a well-understood and frequently employed method of providing incentives. There are concerns, however, that in dynamic settings, incentives may become undermined by the so-called *discouragement effect*.[1] As for losers of earlier stages, winning the overall contest requires efforts beyond the ones necessary for catching up, they are discouraged from fighting on. This in turn allows winners of earlier stages to reduces their efforts, leading to a deterioration of incentives on aggregate. As a consequence of the discouragement effect, innovation may be obstructed from successful completion by an early break through and workers may become demotivated to strive for promotion by an early success of their colleagues.

In this article, we analyze the effect of private information on players' incentives to exert effort in a dynamic contest. Although in many economic settings, private information has a negative impact, in dynamic contests private information turns out to be beneficial. Our starting point is the observation that when players are privately informed about the (common) value of the prize and cannot observe each others efforts, then an early loss has to be taken as good news about the value of the prize. This is because a loss is more likely to happen when the opponent exerted a high level of effort which requires that the opponent attaches a high value to the contest's prize. For example, an early innovation break through may be the consequence of a large R&D effort by a rival company whose private market-research has revealed a prosperous market for the prospective product. Similarly, a rival's success in the early stages of a promotion contest can be understood as the result of his hard work, which was motivated by his positive view about the company's

[1] See Konrad and Kovenock (2009); Ryvkin (2011);Dubey (2013); Fu et al. (2018); Aiche et al. (2019); Fang et al. (2020); Sela and Tsahi (2020);Zhang and Zhou (2016).

career prospects.

In our model two homogeneous players compete in a best-of-three contest by exerting costly efforts in three sequential battles. The winner of the overall contest obtains a prize whose value is uncertain (one or zero) but the same for each player.[2] Each player observes a private imperfect signal (good or bad) that is informative about the prize. Motivated by the above examples, an important assumption of our model is that players are unable to observes their rival's efforts. All they observe is the identity of the winner of each battle.

For most models of dynamic contests, the introduction of imperfect information makes the analysis intractable. Our model lends its tractability from the simplifying assumption that one of the signal-realizations is conclusive about the value of the prize. In particular, assuming that a bad signal can only be received when the prize is zero enables us to focus our analysis on the efforts players exert after receiving a good signal. An important variable of our model is the signal's informativeness, given by the likelihood with which a bad signal is generated when the prize is zero.

As a benchmark, we consider the case, where signals are observed publicly rather than privately. Our main result shows that (expected) aggregate effort, that is individual efforts added across players and battles, is strictly higher under private information than under public information, except for the limiting case where signals are perfectly informative or completely uninformative. Interestingly, while under public information, aggregate effort is independent of the signals' informativeness, under private information, aggregate effort is maximized when signals are of intermediate informativeness. Hence, although our analysis is restricted to a subset of all possible information-structures by our assumption that one signal is conclusive, our model offers insights about the optimal information design in dynamic contest settings.

[2] An alternative but analogue formulation of our model assumes that the contest's prize is certain but that players face uncertainty about the (common) value of their marginal costs of effort. For example, R&D expenditures may depend on a common input (e.g. labor) whose price is uncertain.

2 Model

We consider two homogeneous players engaged in a dynamic best-of-three contest for a single prize. The contest consists of three identical, consecutive battles and the prize is awarded to the first player achieving a total number of two wins. In each battle $t \in \{1, 2, 3\}$, each player $i \in \{1, 2\}$ chooses an effort $e_{it} \geq 0$ at cost $C(e_{it}) = e_{it}$ and wins the battle with probability $\frac{e_{it}}{e_{1t}+e_{2t}}$ when $e_{1t} + e_{2t} \neq 0$ and with probability $\frac{1}{2}$ when $e_{1t} + e_{2t} = 0$. An important assumption of our model is that players observe the outcome of each battle, but not the effort exerted by their opponent.

There is uncertainty about the size of the prize, which can take two values normalized to $V = 0$ and $V = 1$. Both values are assumed to be equally likely. Before the start of the first battle, each player obtains a private informative signal $s_i \in \{B, G\}$ about the value of v. Signals are independent draws from the same conditional probability distribution $Pr(s_i|v)$ specified by the following matrix: The parameter $\sigma \in (0, 1)$ measures the

$Pr(s_i\|v)$	$V = 0$	$V = 1$
$s_i = B$	σ	0
$s_i = G$	$1 - \sigma$	1

informativeness of the players' signals. In particular, for $\sigma \to 1$ players become perfectly informed about the value of the prize, whereas for $\sigma \to 0$ signals become completely uninformative. Note that implicit in this formulation is the assumption that a "bad" signal $s_i = B$ is conclusive, as it can only be received when the prize is zero. This assumption greatly simplifies the analysis because it implies that efforts must be zero after the observation of a bad signal. Hence, our analysis can concentrate on the players' behavior conditional on receiving a "good" signal. In our setting, a symmetric equilibrium can thus be described by a vector of efforts $(e_1^*, e_L^*, e_F^*, e_3^*)$ which players exert after observing $s_i = G$. Here e_1^* and e_3^* denote efforts during the first and the third battle, respectively, whereas e_L^* and e_F^* denote a player's effort in the intermediate battle depending on whether the player has won (leader) or lost (follower) before.[3]

3 Equlibrium

We determine the equilibrium $(e_1^*, e_L^*, e_F^*, e_3^*)$ by backward-induction.

3.1 Battle 3

Suppose that both players have accumulated one win and consider a player's effort choice in battle 3. If the player has obtained a bad signal he knows that there is nothing to be won and will therefore exert zero effort. In contrast, if the player has obtained a good signal, then he must conclude that his opponent has also observed a good signal, since otherwise the opponent would not have exerted effort and could not have won any of the contests before. Hence, conditional on having observed a good signal a player's expected value of the prize is given by

$$V_G \equiv \mathbb{E}[V|s_1 = s_2 = G] = \frac{1}{1 + (1-\sigma)^2} \tag{1}$$

and his effort in the third battle must solve

$$e_3 \in \arg\max_{e_3 \geq 0} \frac{e_3}{e_3 + e_3^*} V_G - e_3. \tag{2}$$

The unique solution to this problem follows from evaluating the corresponding first-order condition at $e_3 = e_3^*$ and is given by

$$e_3^* = \frac{V_G}{4}. \tag{3}$$

The maximized value

$$U_3 \equiv \frac{e_3^*}{e_3^* + e_3^*} V_G - e_3^* = \frac{V_G}{4} \tag{4}$$

represents the players' continuation value from drawing (one win each) after completion of the first two battles.

[3] A player's effort in the third battle is independent of the sequencing of past-outcomes (win-loss, loss-win) because in equilibrium the third battle can be reached only when *both* players have observed a good signal (and hence exert positive efforts), giving players identical beliefs about the value of the prize.

3.2 Battle 2

Consider now the second battle and denote by L (*leader*) and F (*follower*) the winner and the loser of the first battle. Conditional on having observed a good signal, player $i \in \{L, F\}$ must form beliefs about the likelihood with which his opponent j has also observed a good signal. These beliefs are crucial determinants of a player's effort because only when the opponent has observed a good signal there actually exists a prize worth fighting for ($V > 0$) and effort is necessary for winning the second battle ($e_j > 0$).

Assuming that the follower exerted effort in the first battle, he must conclude from losing that battle that his opponent has observed a good signal with certainty. Had his opponent observed a bad signal he would have exerted zero effort and would not have defeated him. In particular, we have

$$Pr(s_j = G | i = F, s_i = G) = 1. \tag{5}$$

In contrast to the follower, the leader does not know whether he won the first battle because he was more lucky or because his opponent failed to provide effort after observation of a bad signal. Moreover, the distinction between these two cases depends on the effort the leader has taken in battle 1. In particular, if the leader chose effort $e_1 > 0$ and the follower adheres to the equilibrium, then the leader would have won the first contest with probability $\frac{e_1}{e_1 + e_1^*}$ in case $s_j = G$ and with probability 1 in case $s_j = B$. Bayesian updating thus implies that from the viewpoint of the leader, the likelihood with which the follower has observed a good signal is given by

$$Pr(s_j = G | i = L, s_i = G) = \frac{1 + (1 - \sigma)^2}{1 + (1 - \sigma)^2 + \frac{e_1 + e_1^*}{e_1} \sigma (1 - \sigma)} \equiv P_2(e_1). \tag{6}$$

It is important to note that

$$Pr(s_j = G | i = F, s_i = G) > Pr(s_j = G | i = L, s_i = G), \tag{7}$$

which means that the follower has a stronger belief than the leader that the prize is worth fighting for. It is in this sense, that losing the first battle "encourages" effort in the second battle.[4]

In equilibrium, effort choices (e_L^*, e_F^*) must satisfy:

$$\begin{aligned} e_L^* &\in \arg\max_{e_L \geq 0} P_2^* \left[\frac{e_L}{e_L + e_F^*} V_G + (1 - \frac{e_L}{e_L + e_F^*}) U_3 \right] - e_L && (8) \\ e_F^* &\in \arg\max_{e_F \geq 0} \frac{e_F}{e_L^* + e_F} U_3 - e_F, && (9) \end{aligned}$$

where we have abbreviated notation by letting

$$P_2^* \equiv P_2(e_1^*) = \frac{1 + (1-\sigma)^2}{1 + (1-\sigma)^2 + 2\sigma(1-\sigma)}. \tag{10}$$

The corresponding first-order conditions

$$\begin{aligned} P_2^*(V_G - U_3) \frac{e_F^*}{(e_L^* + e_F^*)^2} &= 1 && (11) \\ U_3 \frac{e_L^*}{(e_L^* + e_F^*)^2} &= 1 && (12) \end{aligned}$$

have the solution

$$\begin{aligned} e_F^* &= \frac{3}{4} V_G \frac{P_2^*}{(3P_2^* + 1)^2} && (13) \\ e_L^* &= \frac{9}{4} V_G \frac{(P_2^*)^2}{(3P_2^* + 1)^2}. && (14) \end{aligned}$$

The probability with which the players draw after the second battle, making the third battle necessary, is given by

$$\frac{e_F^*}{e_F^* + e_L^*} = \frac{1}{1 + 3P_2^*}. \tag{15}$$

For future reference, note that the probability of a draw is inverse U-shaped with a maximum at

$$\sigma^{draw} \equiv 2 - \sqrt{2} \in (0, 1). \tag{16}$$

[4]The fact that a deviation from e_1^* to $e_1 \neq e_1^*$ influences the informativeness of the first battle's outcome will have an effect for the determination of the equilibrium effort level e_1^* in Section 3.3.

Off-equilibrium, that is, after deviating to $e_1 \neq e_1^*$ during the first battle, the leader will update his belief to $P_2(e_1)$ rather than P_2^* and hence choose the effort e_L that solves (11) with P_2^* substituted by $P_2(e_1)$. Off-equilibrium, the leader will therefore choose

$$e_L(e_1) = \sqrt{P_2(e_1)(V_G - U_3)e_F^*} - e_F^* \tag{17}$$

with e_F^* given by (13).

For our analysis of battle 1 contained in the subsequent section it is useful to define the continuation values of entering battle 2 as the leader or the follower, conditional on the opponent having observed a good or a bad signal. These continuation values are given by

$$\begin{aligned} U_L^G(e_1) &\equiv U_3 + \frac{e_L(e_1)}{e_L(e_1) + e_F^*}(V_G - U_3) - e_L(e_1) && (18) \\ U_L^B(e_1) &\equiv -e_L(e_1) && (19) \\ U_F^G &\equiv \frac{e_F^*}{e_F^* + e_L^*} U_3 - e_F^* && (20) \\ U_F^B &\equiv -e_F^*. && (21) \end{aligned}$$

3.3 Battle 1

Finally, consider the players' behavior in battle 1. A player who observes a bad signal will not exert any effort. A player who observes a good signal will believe that his opponent also observed a good signal with probability

$$P_1 = \frac{1 + (1-\sigma)^2}{2 - \sigma}. \tag{22}$$

The effort a player chooses in the first battle after observing a good signal must solve:

$$e_1^* \in \arg\max_{e_1 > 0} P_1 [\frac{e_1}{e_1 + e_1^*} U_L^G(e_1) + \frac{e_1^*}{e_1 + e_1^*} U_F^G] + (1 - P_1) U_L^B(e_1) - e_1. \tag{23}$$

Note that here we have made use of the fact that, conditional on the opponent having observed a bad signal, a player with a good signal cannot become the follower in battle

2 if he exerts effort (no matter how small) in battle 1. The equilibrium value e_1^* can be obtained from the first order condition corresponding to (23) evaluated at $e_1 = e_1^*$:

$$P_1 \frac{U_L^G(e_1^*) - U_F^G}{4e_1^*} + \frac{d}{de_1}\left[\frac{P_1}{2}U_L^G(e_1) + (1-P_1)U_L^B(e_1)\right]|_{e_1=e_1^*} = 1. \tag{24}$$

Note that

$$\frac{P_1}{2}U_L^G(e_1) + (1-P_1)U_L^B(e_1) = \frac{P_2^*[\frac{e_L(e_1)}{e_L(e_1)+e_F^*}V_G + (1-\frac{e_L(e_1)}{e_L(e_1)+e_F^*})U_3] - e_L(e_1)}{\frac{1}{2}P_1 + 1 - P_1} \tag{25}$$

and that the numerator in (25) is identical to the leader's objective function in battle 2 when his belief equals the equilibrium value P_2^*. It thus follows from the envelope theorem that

$$\frac{d}{de_1}\left[\frac{P_1}{2}U_L^G(e_1) + (1-P_1)U_L^B(e_1)\right]|_{e_1=e_1^*} = 0. \tag{26}$$

Hence the first order condition (24) simplifies to

$$P_1 \frac{U_L^G(e_1^*) - U_F^G}{4e_1^*} = 1 \Leftrightarrow P_1 \frac{\frac{e_L^*}{e_L^*+e_F^*}V_G - (e_L^* - e_F^*)}{4e_1^*} = 1. \tag{27}$$

The effect of an increase in e_1 is to raise a player's likelihood of winning the first battle by $\frac{1}{4e_1^*}$. This raises the player's chance of securing an early victory and reduces the risk of an early victory by the opponent, leading an overall benefit of $V_G - U_3 + U_3 = V_G$. As both of these events require that the respective leader wins the second battle, V_G is multiplied with the leader's probability of winning $\frac{e_L^*}{e_L^*+e_F^*}$. By making a player more likely to become the leader rather than the follower, an increase in e_1 also induces additional effort costs $e_L^* - e_F^* > 0$. As

$$\frac{e_L^*}{e_L^*+e_F^*}V_G - (e_L^* - e_F^*) = V_G \frac{3P_2^*(9P_2^*+5)}{4(3P_2^*+1)^2} > 0, \tag{28}$$

the overall effect is positive and must be balanced by the marginal cost of first stage effort (the right hand side of (24)) leading to

$$e_1^* = V_G \frac{3P_1P_2^*(9P_2^*+5)}{16(3P_2^*+1)^2}. \tag{29}$$

3.4 Aggregate effort

Aggregating efforts across battles and players, we obtain the following expression for expected aggregate effort:

$$E^* = Pr(s_1 = s_2 = G)(2e_1^* + e_L^* + e_F^* + \frac{e_F^*}{e_F^* + e_L^*} \cdot 2e_3^*) + Pr(s_1 \neq s_2)(e_1^* + e_L^*). \tag{30}$$

With probability

$$Pr(s_1 = s_2 = G) = \frac{1}{2}[1 + (1 - \sigma)^2] \tag{31}$$

both players obtain a good signal and therefore exert the corresponding efforts in battles 1 and 2. Efforts in battle 3 are exerted only if it is reached which happens when the follower wins the second battle, i.e. with probability $\frac{e_F^*}{e_F^*+e_L^*}$. With probability

$$Pr(s_1 \neq s_2) = \frac{1}{2} \cdot 2\sigma(1 - \sigma) \tag{32}$$

one player receives a good signal while the other player receives a bad signal. In this case only the player with the good signal exerts efforts and wins the contest already after two battles.

In the Appendix we proof the following result:

Proposition 1. *Expected aggregate effort $E^*(\sigma)$ is maximized when information levels the playing field in the intermediate battle, that is for $\sigma = \sigma^{draw} \in (0, 1)$.*

Proof: See Appendix.

Proposition 1 is depicted in Figure 1. Aggregate effort is inverse U-shaped, with a maximum at the place where the likelihood that the second battle is one by the follower rather than by the leader is highest. Aggregate incentives are maximized when the players' information makes their incentives to compete in the second battle as similar as possible. For this purpose, the encouraging news contained in a first battle loss is sought to counter the discouraging effect of a lag in the number of accumulated wins, thereby balancing the follower's and the leader's incentives to fight.

4 Benchmark: Public signals

In order to understand the effect that private information has on the players' incentives to exert effort, this section considers as a benchmark the case where information is *public*. In particular, the following analysis assumes that signals (s_1, s_2) are observed publicly, by both players, rather than privately.

As a bad signal is conclusive, under public information a player will exert effort only after observing $(s_1, s_2) = (G, G)$. The players' expectations of the contest's prize is then given by V_G and the analysis of battle 3 is the same as under private information (where players conclude that their opponent must have observed a good signal since otherwise the last battle would not have been reached). In particular

$$e_3^{PUB} = e_3^* = \frac{V_G}{4}. \tag{33}$$

In battle 2, players no longer differ in their beliefs about their opponent's signal. The equilibrium efforts under public information can therefore be obtained by setting $P_2 = 1$ in (8)–(14) and are given by

$$\begin{aligned} e_F^{PUB} &= \frac{3V_G}{64} && (34) \\ e_L^{PUB} &= \frac{9V_G}{64}. && (35) \end{aligned}$$

Under public information the leader wins also the second battle with probability $\frac{3}{4}$. Hence, the corresponding maximized payoffs of the leader and the follower are given by

$$\begin{aligned} U_L &= \frac{3}{4}V_G + \frac{1}{4}\frac{V_G}{4} - e_L^{PUB} = \frac{43}{64}V_G && (36) \\ U_F &= \frac{1}{4}\frac{V_G}{4} - e_F^{PUB} = \frac{1}{64}V_G && (37) \end{aligned}$$

Finally, in battle 1, equilibrium effort must solve

$$e_1^{PUB} \in \arg\max_{e_1 \geq 0} U_F + \frac{e_1}{e_1 + e_1^{PUB}}(U_L - U_F) - e_1 \tag{38}$$

and is hence given by

$$e_1^{PUB} = \frac{U_L - U_F}{4} = \frac{21}{128} V_G. \tag{39}$$

Expected aggregate effort under public information is then given by

$$E^{PUB} = Pr(s_1 = s_2 = G)(2e_1^{PUB} + e_L^{PUB} + e_F^{PUB} + \frac{1}{4} \cdot 2e_3^{PUB}) = \frac{41}{128} \tag{40}$$

Proposition 2. *Expected aggregate effort is higher when signals are observed privately rather than publicly. This result holds independently of the signals' quality, i.e.* $E^*(\sigma) > E^{PUB}$ *for all* $\sigma \in (0, 1)$.

Proof: See Appendix.

Figure 1 depicts aggregate effort in dependence of the informativeness σ of the players' signals. While aggregate effort is independent of σ when signals are public, under private information aggregate effort is inverse U-shaped. Proposition 2 emphasizes the important

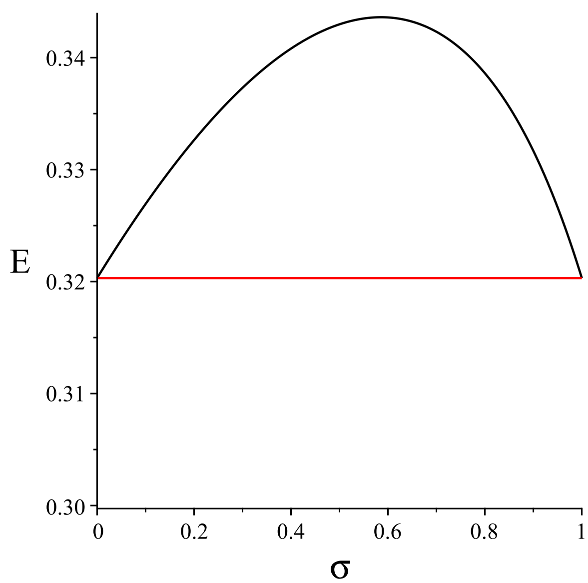

Figure 1: **Aggregate Effort**: Comparison of aggregate effort under private (black) and public information (red), in dependence of the quality σ of the players' signals. $\sigma = 1$ and $\sigma = 0$ correspond to the cases where signals are perfectly informative or not informative at all.

role of private information in dynamic battles. In particular, it shows that withholding information about a contest's prize can be useful only when players may form differing beliefs about the prospects of winning.

5 Static contest

In order to understand the origin of Proposition 1 it is useful to consider the static analogue of our dynamic contest setting. For this purpose, suppose that the contest consists of only one battle and the prize V is awarded to the winner of that battle. Under private information equilibrium effort must solve

$$e_S^* \in \arg\max_{e_S \geq 0} P_1 \frac{e_S}{e_S + e_S^*} V_G - e_S \tag{41}$$

and is therefore given by

$$e_S^* = \frac{P_1 V_G}{4} = \frac{1}{4} \frac{1}{2 - \sigma}. \tag{42}$$

Given that each player receives a good signal with probability

$$Pr(s_i = G) = 1 - \frac{\sigma}{2}, \tag{43}$$

expected aggregate effort is

$$E_S^* = 2Pr(s_i = G) e_S^* = \frac{1}{4}. \tag{44}$$

Under public information both signals are observed by both players and players will exert effort only when both signals are good. Equilibrium effort solves

$$e_S^{PUB} \in \arg\max_{e_S \geq 0} \frac{e_S}{e_S + e_S^{PUB}} V_G - e_S \tag{45}$$

and is given by

$$e_S^{PUB} = \frac{V_G}{4}. \tag{46}$$

Expected aggregate effort under public information is

$$E_S^{PUB} = 2Pr(s_1 = s_2 = G)e_S^{PUB} = \frac{1}{4} \tag{47}$$

Note that in the static contest, expected aggregate effort under private information is the same as expected aggregate effort under public information. Whether signals are private or public has no influence on aggregate incentives in a static setting. Hence it must be the dynamic nature of our best-of-three contest which explains the increase in aggregate incentives due to private information. One might have thought that private information raises aggregate effort because it allows a player to exert effort even when his opponent has observed a bad signal. However, this reasoning applies equally well in the static and the dynamic setting, and as aggregate effort in the static setting remains unaffected, cannot be the reason behind the increase in aggregate effort in the dynamic setting.

In order to understand why private information may increase aggregate incentives in dynamic contests, we will now have a closer look at the individual efforts in each of the three battles.

6 The encouragement effect

So why does private information increase aggregate incentives? In order to shed light on this question, let us consider each battle in separation. As efforts in battle 3 are identical under private and public information, we can concentrate our analysis on battles 1 and 2.

Consider battle 2 first. Winning battle 2 is less beneficial for the follower than for the leader, as the follower is required to exert additional effort to win the overall contest in battle 3. This fact has become known as the *discouragement effect*. As we have seen in Section 3.2 private information makes winning the first battle bad news about the contest's prize, thereby dampening the leader's incentive to exert effort. As a consequence, the follower becomes encouraged to exert higher effort, bringing efforts levels in battle 2 closer to each other. In particular, we have the following result:

Proposition 3 (Encouragement)**.** *When signals are observed privately rather than publicly, the effort-differential between the leader and the follower is reduced, i.e.* $0 < e_L^* - e_F^* < e_L^{PUB} - e_F^{PUB}$*. As a consequence, the final battle is reached with higher probability when signals are private.*

Proof: See Appendix.

Private information "levels the playing field" in that it endows the disadvantaged player with a stronger belief in the value of fighting. The downside of this *encouragement effect* is that players' will attach less value to becoming the leader. In particular, we also have the following result:

Proposition 4. *When signals are observed privately rather than publicly, players invest less effort to become the leader, i.e.* $e_1^* < e_1^{PUB}$.

Proof: See Appendix.

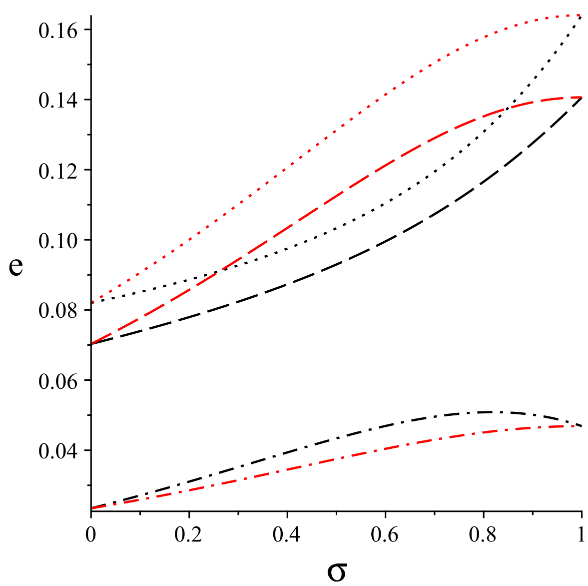

Figure 2: **Battle Efforts**: Comparison of efforts e_1 (dotted) e_L (dashed), and e_F (dash-dotted) under private (black) and public information (red), in dependence of the quality σ of the players' signals. $\sigma = 1$ and $\sigma = 0$ correspond to the cases where signals are perfectly informative or not informative at all.

Figure 2 depicts efforts in battles 1 and 2 in dependence of the informativeness σ of players signals. Private information raises the follower's effort while reducing the leader's effort in battle 2. This raises the likelihood of a draw after battle 2 which reduces efforts in battle 1.

7 The last shall be first?

In our model, the follower is discouraged by the fact that he has fallen behind but becomes encouraged by the "good news" contained in his first-battle loss. The overall effect is that his effort falls short of the leader's effort, i.e. $e_F^* < e_L^*$. We now extend our model, in order to show, that, more generally, the encouragement effect may be strong enough to overcome the discouragement effect. In particular, we will show that, in the spirit of Matthews's quote, the loser of the first stage is more likely to be the winner of the second stage, i.e. $e_F^* > e_L^*$, if the contest's outcome is sufficiently noisy.

For this purpose, we assume that in each battle t, player i wins with probability $\frac{e_{it}^r}{e_{it}^r+e_{jt}^r}$. Following Tullock, the parameter $r \in (0,2)$ measures how sensitive a player's probability of winning is with respect to his own effort. Decreasing r makes the battle's outcome less sensitive with respect to efforts, or, equivalently, more random.

In the last battle, if it is reached, equilibrium effort must solve

$$e_3^* \in \arg\max_{e_3 \geq 0} \frac{e_3^r}{e_3^r + (e_3^*)^r} V_G - e_3$$

and is therefore given by

$$e_3^*(r) = \frac{r}{4} V_G.$$

With the continuation value $U_3(r) = \frac{2-r}{4} V_G$ of reaching the last battle, in the second battle, equilibrium efforts must solve:

$$e_L^* \in \arg\max_{e_L \geq 0} P_2^* \left[\frac{e_L^r}{e_L^r + (e_F^*)^r} V_G + (1 - \frac{e_L^r}{e_L^r + (e_F^*)^r}) U_3(r) \right] - e_L \tag{48}$$

$$e_F^* \in \arg\max_{e_F \geq 0} \frac{e_F^r}{(e_L^*)^r + e_F^r} U_3(r) - e_F. \tag{49}$$

From the corresponding first order conditions

$$P_2^* \frac{r e_L^{r-1} e_F^r}{(e_L^r + e_F^r)^2}(1 - \frac{2-r}{4})V_G = 1 = \frac{r e_F^{r-1} e_L^r}{(e_L^r + e_F^r)^2}\frac{2-r}{4}V_G.$$

it follows that

$$\frac{e_F^*(r)}{e_L^*(r)} = \frac{2-r}{(2+r)P_2^*}.$$

As this expression is decreasing in r and converges to $1/P_2^* > 1$ for $r \to 0$, we can state the following result:

Proposition 5. *The encouragement effect overcomes the discouragement effect when battle outcomes are sufficiently noisy, i.e.* $e_F^*(r) > e_L^*(r)$ *for all* $r \in (0, \bar{r})$*, with* $0 < \bar{r} < 1$.

Proof: See Appendix.

Note that, since $P_2(\sigma)$ is U-shaped, $\bar{r}(\sigma)$ is inverse U-shaped with $\lim_{\sigma \to 0} \bar{r}(\sigma) = \lim_{\sigma \to 1} \bar{r}(\sigma) = 0$, i.e. more noise is required to make the last become first, when signals are very informative or very uninformative, i.e. when the privacy of information becomes irrelevant.

8 Conclusion

This paper has identified an encouragement effect in dynamic contests where players have private information about the contest's prize. Losers of early stages become encouraged by the fact that losing conveys good news about their opponent's effort and hence the value of the contest's prize. We have shown that due to the encouragement effect, aggregate incentives are maximized when information about the contest's prize is private rather than public and imperfect rather than perfect. Our model thereby provides insights about the optimal information design in dynamic contests.

Appendix: Proofs

Proof of Propositions 1 and 2:

Since $V_G = \frac{1}{1+(1-\sigma)^2}$, and $P_2^* V_G = \frac{1}{2-\sigma^2}$, then, after some algebraic manipulations

$$E^*(\sigma) = \frac{1}{16(3P_2^*+1)^2}\{3P_1P_2^*(9P_2^*+5) + 6P_2^*(3P_2^*+1) + 4(3P_2^*+1) + \frac{\sigma(1-\sigma)}{2-\sigma^2}[3P_1(9P_2^*+5) + 36P_2^*]\}.$$

Thus,

$$\begin{aligned}
& E^*(\sigma) \geq E^{PUB} = \frac{41}{128} \\
\Leftrightarrow \quad & 3P_1P_2^*(9P_2^*+5) + 6P_2^*(3P_2^*+1) + 4(3P_2^*+1) + \\
& \quad + \frac{\sigma(1-\sigma)}{2-\sigma^2}[3P_1(9P_2^*+5) + 36P_2^*] \geq \frac{41}{8}(3P_2^*+1)^2
\end{aligned}$$

Using $\frac{\sigma(1-\sigma)}{2-\sigma^2} = \frac{P_2^{*2}}{P_1}(1-P_1)$, the previous inequality holds iff

$$\begin{aligned}
& 3P_2^*(9P_2^*+5) + 6P_2^*(3P_2^*+1) + 4(3P_2^*+1) + 36\frac{P_2^{*2}}{P_1}(1-P_1) \geq \frac{41}{8}(3P_2^*+1)^2 \\
\Leftrightarrow \quad & 27P_2^* + 3 + 36\frac{P_2^{*2}}{P_1} \geq \frac{33}{8}[9P_2^{*2} + 6P_2^* + 1] \\
\Leftrightarrow \quad & P_2^*[18 + 288\frac{P_2^*}{P_1} - 297P_2^*] \geq 9.
\end{aligned}$$

Since $P_2^* = \frac{1+(1-\sigma)^2}{2-\sigma^2}$ and $P_1 = \frac{1+(1-\sigma)^2}{2-\sigma}$, the left-hand side is inverse U-shape with a maximum at $\sigma = 2-\sqrt{2}$ and a minimum at $\sigma = \{0,1\}$, in which case the left-hand side is equal to 9. ■

Proof Proposition 3:

Using the equations (13), (14), (34), and (35),

$$\begin{aligned}
& e_L^*(\sigma) - e_F^*(\sigma) \leq e_L^{PUB} - e_F^{PUB} \\
\Leftrightarrow \quad & \frac{3V_GP_2^*}{4}\frac{3P_2^*-1}{(3P_2^*+1)^2} \leq \frac{6V_G}{64} \\
\Leftrightarrow \quad & 15P_2^{*2} - 14P_2^* \leq 1,
\end{aligned}$$

which holds for any $P_2^* \in [0, 1]$. ■

Proof Proposition 4:

Using equations (29) and (39):

$$
\begin{aligned}
& e_1^* \leq e_1^{PUB} \\
\Leftrightarrow \quad & \frac{P_1 P_2^* (9P_2^* + 5)}{(3P_2^* + 1)^2} \leq \frac{7}{8} \\
\Leftrightarrow \quad & 9P_1 + 5\frac{P_1}{P_2^*} \leq \frac{7}{8}(3 + \frac{1}{P_2^*})^2 \\
\Leftrightarrow \quad & 9\frac{1 + (1-\sigma)^2}{2-\sigma} + 5\frac{2-\sigma^2}{2-\sigma} \leq \frac{7}{8}(3 + \frac{2-\sigma^2}{1 + (1-\sigma)^2})^2.
\end{aligned}
$$

The left-hand side is U-shaped with maximums at $\sigma = \{0, 1\}$ when it gets a value of 14. The right-hand side is inverse U-shaped with minimums also at $\sigma = \{0, 1\}$ when its value is 14. ■

Proof Proposition 5:

The expression $\frac{2-r}{(2+r)P_2^*}$ is decreasing and continuous in r, with $1/3P_2^* < 1$ when $r \to 1$ and $1/P_2^* > 1$ when $r \to 0$. ■

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